# Math 312 Lectures 4 and 5 Second Order Differential Equations; Nondimensional Equations 

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We introduce second order differential equations, and then discuss the technique of nondimensionalizing a differential equation ${ }^{1}$.

Second order differential equations. A general form of a second order differential equation is

$$
\begin{equation*}
\frac{d^{2} y}{d t^{2}}=f\left(t, y, \frac{d y}{d t}\right) \tag{1}
\end{equation*}
$$

It is an equation that relates a function to its first and second derivatives. For example,

$$
\begin{equation*}
\frac{d^{2} y}{d t^{2}}=-3 \frac{d y}{d t}-2 y \tag{2}
\end{equation*}
$$

says we want a function $y(t)$ with the property that its second derivative is equal to the given linear combination of the function and its first derivative for all $t$. You should verify that $y(t)=e^{-t}$ satisfies this equation, as does $y(t)=e^{-2 t}$.

A trivial second order differential equation is

$$
\begin{equation*}
\frac{d^{2} y}{d t^{2}}=0 . \tag{3}
\end{equation*}
$$

This one we can solve by simply integrating twice ${ }^{2}$ :

$$
\begin{equation*}
\frac{d y}{d t}=A, \quad y=A t+B \tag{4}
\end{equation*}
$$

where $A$ and $B$ are constants of integration. $y=A t+B$ is a solution for any constants $A$ and $B$. For a first order differential equation, we know we need an initial condition $y(0)=y_{0}$ to determine the value of the arbitrary constant that shows up in the solution. For a second order differential

[^0]equation, there are generally two arbitrary constants, so we need two initial conditions: $y(0)=y_{0}$, $y^{\prime}(0)=v_{0}$. So a general form for a second order initial value problem is
\[

$$
\begin{equation*}
\frac{d^{2} y}{d t^{2}}=f\left(t, y, \frac{d y}{d t}\right) \quad y(0)=y_{0}, \quad y^{\prime}(0)=v_{0} . \tag{5}
\end{equation*}
$$

\]

(With these initial conditions, you should verify that the solution to the trivial differential equation $\frac{d^{2} y}{d t^{2}}=0$ is $y(t)=v_{0} t+x_{0}$.) In problems where $y(t)$ represents the position of an object, $y_{0}$ is the initial position, and $v_{0}$ is the initial velocity.

The projectile problem. We consider the problem of determining the height of an object that is launched vertically from the surface of the earth with initial speed $v_{0}$. Let $t$ be the time, measured from the instant that the object is launched, let $x(t)$ be the height of the object above the surface of the earth, let $g$ be the gravitational acceleration, and let $R$ be the radius of the earth. Newton's laws may be used to derive the following differential equation for $x(t)$ :

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}=-\frac{g R^{2}}{(x+R)^{2}}, \quad x(0)=0, \quad x^{\prime}(0)=v_{0} . \tag{6}
\end{equation*}
$$

If we were going to perform computations with this equation and compare the solutions to actual experiments, we would need to work with a consistent set of units. For example, we might measure time in seconds (sec), distance in meters (m), and mass in kilograms (kg). In this case, the units of $g$ are $\mathrm{m} / \mathrm{sec}^{2}$. The quantities time, length and mass are dimensions. For our equation to make sense, we must measure all dimensions with consistent units. Note that the dimension of a variable is an inherent property of the variable, but the units are something we can choose. For example, $x$ is a length, but we might choose meters, miles, or even furlongs for its units. In the following, we will want to indicate the dimensions of all the variables and parameters in a problem. We'll use the symbols $\mathcal{L}$ for length, $\mathcal{T}$ for time, and $\mathcal{M}$ for mass.

The idea is to measure our variables in "units" that are instrinsic to the problem. Units such as kilometers or miles are arbitrary. The following procedure will let us choose units that can simplify the problem. Specifically, this procedure usually reduces the number of parameters in the problem.

## Procedure for Nondimensionalizing a Differential Equation.

1. List all the variables and parameters along with their dimensions.
2. For each variable, say $x$, form a product (or quotient) $p$ of parameters that has the same dimensions as $x$, and define a new variable $y=x / p . y$ is a "dimensionless" variable. Its numerical value is the same no matter what system of units is used.
3. Rewrite the differential equation in terms of the new variables.
4. In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters. (This will typically result in fewer parameters.)

We'll apply this procedure to the projectile problem, but first we point out an important aspect of step 3. Time $t$ is one of the variables in the problem (usually we use it as the independent variable), so in step 2 we will create a nondimensional version of this variable, say $\tau$. Since the differential equation has derivatives with respect to $t$, and we want the new equation to be expressed

| Variable or Parameter | Meaning | Dimension |
| :---: | :--- | :--- |
| $t$ | time since the launch of the object | $\mathcal{T}$ |
| $x$ | distance from the surface of the earth | $\mathcal{L}$ |
| $g$ | gravitational constant | $\mathcal{L} \mathcal{T}^{-2}$ |
| $R$ | radius of the earth | $\mathcal{L}$ |
| $v_{0}$ | initial velocity | $\mathcal{L} \mathcal{T}^{-1}$ |

Table 1: The list of variables and parameters for the projectile problem, along with their dimensions. $\mathcal{T}$ means time and $\mathcal{L}$ means length.
in terms of $\tau$, we will have to use the chain rule to convert from $t$ to $\tau$. Suppose, for example, we have the dimensional variables $t$ and $x(t)$, and we define nondimensional variables $\tau=t / T$ and $y=x / P$. How do we express $\frac{d x}{d t}$ and $\frac{d^{2} x}{d t^{2}}$ in terms of $\tau$ and $y$ ? First, write $y=x / P$ a little more carefully as

$$
\begin{equation*}
y(\tau)=\frac{x(T \tau)}{P}=\frac{x(t)}{P} \tag{7}
\end{equation*}
$$

or

$$
\begin{equation*}
x(t)=P y(t / T)=P y(\tau) \tag{8}
\end{equation*}
$$

Now take the derivative with respect to $t$ on both sides. We will have to use the chain rule on the right.

$$
\begin{equation*}
\frac{d x(t)}{d t}=P \frac{d}{d t}(y(\tau))=P \frac{d y}{d \tau} \frac{d \tau}{d t}=\frac{P}{T} \frac{d y}{d \tau} \tag{9}
\end{equation*}
$$

I included the $t$ and $\tau$ arguments in the first few expressions to remind you of what the arguments of $x$ and $y$ are, but from here on, I will suppress the arguments. The last equality in (9) comes from $\frac{d \tau}{d t}=1 / T$, since $\tau=t / T$. We also find

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}=\frac{P}{T} \frac{d^{2} y}{d \tau^{2}} \frac{d \tau}{d t}=\frac{P}{T^{2}} \frac{d^{2} y}{d \tau^{2}} \tag{10}
\end{equation*}
$$

Higher derivatives can be found the same way.
Once we understand how this works, we can take advantage of a formal shortcut. To express $\frac{d x}{d t}$ or $\frac{d^{2} x}{d t^{2}}$ in terms of $\tau$ and $y$, where $t=T \tau$ and $x=P y$, simply substitute the variables in the expression for the derivative:

$$
\begin{equation*}
\frac{d x}{d t}=\frac{d(P y)}{d(T \tau)}=\frac{P}{T} \frac{d y}{d \tau} \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}=\frac{d^{2}(P y)}{d(T \tau)^{2}}=\frac{P}{T^{2}} \frac{d^{2} y}{d \tau^{2}} \tag{12}
\end{equation*}
$$

This formal procedure seems fishy at first, but it is really just a shortcut for the chain rule.
We'll now apply the nondimensionalization procedure to the projectile problem.
Step 1. Table 1 shows the result of step 1. (I've also included the meaning of each variable and parameter in the list.)
Step 2. The variable $t$ has dimension $\mathcal{T}$, so we must find a combination of the parameters that also has dimension $\mathcal{T}$. We see that $R / v_{0}$ is one such combination. We define

$$
\begin{equation*}
\tau=\frac{t}{\left(R / v_{0}\right)}=\frac{v_{0} t}{R} \tag{13}
\end{equation*}
$$

The variable $x$ has dimension $\mathcal{L}$, and so does $R$, so we define

$$
\begin{equation*}
y=\frac{x}{R} \tag{14}
\end{equation*}
$$

Step 3. We now express (6) in terms of the dimensionless variables $\tau$ and $y$. We have $t=\left(R / v_{0}\right) \tau$ and $x=R y$. Then, by using the shortcut discussed earlier, we have

$$
\begin{equation*}
\frac{d x}{d t}=\frac{d(R y)}{d\left(\left(R / v_{0}\right) \tau\right)}=v_{0} \frac{d y}{d \tau} \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d^{2} x}{d t^{2}}=\frac{d^{2}(R y)}{d\left(\left(R / v_{0}\right) \tau\right)^{2}}=\frac{R}{\left(R / v_{0}\right)^{2}} \frac{d^{2} y}{d \tau^{2}}=\frac{v_{0}^{2}}{R} \frac{d^{2} y}{d \tau^{2}} \tag{16}
\end{equation*}
$$

Also note that when we substitute $x=R y$ into the right side of the differential equation in (6), the $R$ factors in the numerator and denominator cancel. To convert the initial conditions, we use (14) to obtain $y(0)=x(0) / R=0$, and we use (15) to obtain $\frac{d y}{d \tau}(0)=\left(\frac{d x}{d t}(0)\right) / v_{0}=1$. The result of all this is the new equation

$$
\begin{equation*}
\frac{v_{0}^{2}}{R} \frac{d^{2} y}{d \tau^{2}}=-\frac{g}{(y+1)^{2}}, \quad y(0)=0, \quad y^{\prime}(0)=1 \tag{17}
\end{equation*}
$$

Step 4. Now we multiply both sides of the differential equation by $R / v_{0}^{2}$, and define $\beta$ as

$$
\begin{equation*}
\beta=\frac{g R}{v_{0}^{2}} \tag{18}
\end{equation*}
$$

Note that $\beta$ is dimensionless. So our final, nondimensional problem is

$$
\begin{equation*}
\frac{d^{2} y}{d \tau^{2}}=-\frac{\beta}{(y+1)^{2}}, \quad y(0)=0, \quad y^{\prime}(0)=1 \tag{19}
\end{equation*}
$$

Instead of three parameters, we have just one, and everything is dimensionless. This is, in fact, a general result. When an equation is nondimensionalized, new parameters can be defined such that the equation depends only on the new parameters, which are all dimensionless.

The fact that the new variables and parameters are all dimensionless means that the equation does not change if we change our coordinates, say from miles and hours to meters and seconds. Also, the fact that we ended up with just one parameter means that the original three parameters ( $g, R$, and $v_{0}$ ) did not have independent effects on the behavior. Any combinations of $g, R$ and $v_{0}$ that result in the same value of $\beta$ will result in the same behavior of the solution.

Let's go back and look at step 2 again. We defined $y=x / R$. This amounts to choosing the radius of the earth $R$ as our fundamental unit of length. Given the nature of the problem, this is a "natural" or "intrinsic" length scale, as opposed to miles or meters, which are completely arbitrary.

We also defined $\tau=\frac{t}{R / v_{0}}$. Is there a natural or intrinsic meaning of $R / v_{0}$ ? You may recall that if an object moves at a constant velocity $v_{0}$, the distance that it travels in time $T$ is $v_{0} T$. On the other hand, if the object travels a distance $R$ with constant velocity $v_{0}$, the time required is $R / v_{0}$. Thus, the "meaning" of $R / v_{0}$ is the time it would take an object to travel the radius of the earth if it were moving at the constant speed $v_{0}$. Unlike seconds or hours, which are arbitrary, $R / v_{0}$ provides a unit of time that is defined in terms of parameters in the problem; it is an intrinsic time scale.


[^0]:    ${ }^{1}$ Parts of these notes are based heavily on Chapter 6 of Mathematics Applied to Deterministic Problems in the Natural Sciences by C. C. Lin and L. A. Segel (SIAM, 1988).
    ${ }^{2}$ Note that we can not solve (2) by simply integrating. If we integrate once, we obtain $\frac{d y}{d t}=-3 y-2 \int y(t) d t$. Since we don't know what $y(t)$ is (after all, $y(t)$ is what we are trying to find), we can not evaluate $\int y(t) d t$. Except for trivial cases (such as (3)), integrating both sides of a differential equation transforms it into a new problem, but does not solve it.

