Math 312 Lecture 2
An Example from Economics: The Solow Growth Model

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In this lecture we look at a model from macroeconomics. Let K be the capital,[1] L the labor, and Q the production output of an economy. We are interested in a *dynamic* problem, so $K(t)$, $L(t)$ and $Q(t)$ are all functions of time, but we will suppress the t argument. In elementary economics, one learns that a common assumption is that Q can be expressed as function of K and L:

$$Q = f(K, L). \tag{1}$$

We assume that f has, using economics terminology, *constant returns to scale*. Mathematically, this means that multiplying K and L by the same amount results in Q being multiplied by the same amount. That is, for any constant b,

$$f(bK, bL) = bf(K, L).$$

For example, the Cobb-Douglas function $f(K, L) = K^{1/3}L^{2/3}$ satisfies this assumption.

We make two more assumptions. We assume that a constant proportion of Q is invested in capital. This means that the *rate of change* of K is proportional to Q:

$$\frac{dK}{dt} = sQ, \tag{2}$$

where $s > 0$ is the proportionality constant. We also assume that the labor force is growing according to the equation

$$\frac{dL}{dt} = \lambda L, \tag{3}$$

where $\lambda > 0$ is the growth rate. (As you know, this is a simple first order equation for L which we can solve to find $L = L_0 e^{\lambda t}$.)

Now we will combine equations (1), (2), and (3) to obtain a first order differential equation. First we express the relation $Q = f(K, L)$ in a different form. Because f has constant returns to scale, we can write

$$Q = f(K, L) = f\left(L\frac{K}{L}, L\right) = Lf\left(\frac{K}{L}, 1\right) = Lg(k), \tag{4}$$

or

$$\frac{Q}{L} = g(k) \tag{5}$$

[1] *Capital* includes things that are owned to be used in production, such as buildings and manufacturing equipment.

where we have introduced a new variable $k = K/L$ (so k is the *ratio* of capital to labor), and we have defined a new function $g(k) = f(k, 1)$. By differentiating the relation $K = kL$ we obtain

$$\frac{dK}{dt} = k\frac{dL}{dt} + \frac{dk}{dt}L \tag{6}$$

so

$$\begin{aligned} \frac{dk}{dt} &= \frac{1}{L}\left(\frac{dK}{dt} - \frac{dL}{dt}k\right) \\ &= \frac{1}{L}(sQ - \lambda Lk) \\ &= s\frac{Q}{L} - \lambda k \\ &= sg(k) - \lambda k. \end{aligned} \tag{7}$$

Thus we have the *Solow Growth Model*[2]

$$\frac{dk}{dt} = sg(k) - \lambda k \tag{8}$$

which models the growth of the ratio of capital to labor under the assumptions given earlier.

As an example, let's take the production function to be a Cobb-Douglas function $f(K, L) = K^{1/3}L^{2/3}$. Then $g(k) = f(k, 1) = k^{1/3}$, and the differential equation for k is

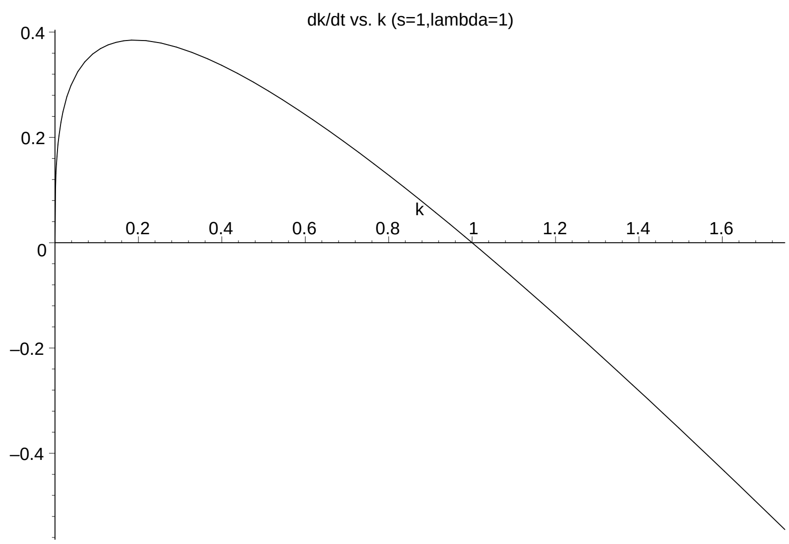
$$\frac{dk}{dt} = sk^{1/3} - \lambda k. \tag{9}$$

By solving

$$sk^{1/3} - \lambda k = 0,$$

we find the equilibrium solutions to be $k = 0$ or $k = (s/\lambda)^{3/2}$.

The following plot shows the graph of $\frac{dk}{dt}$ versus k when $s = 1$ and $\lambda = 1$.



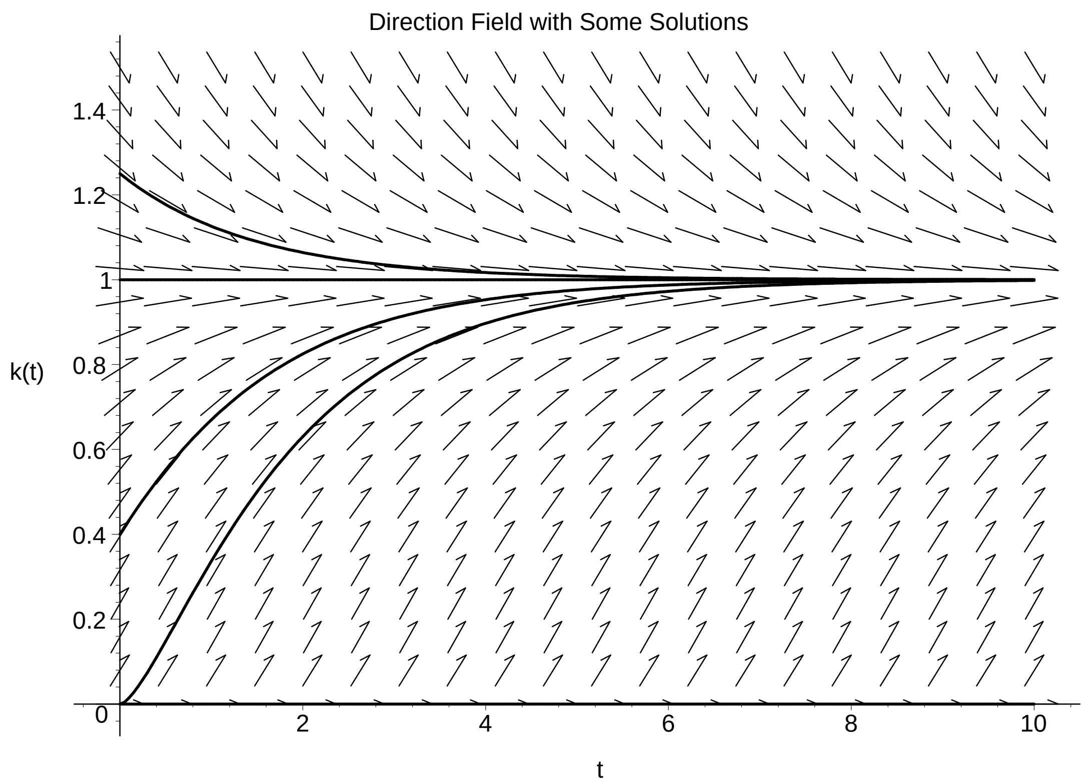
[2]Robert M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, February, 1956, pp. 65–94.

Changing λ or s will change the scale (and the numerical value of the non-zero equilibrium), but the graph of dk/dt versus k will always have the same qualitative shape as the graph shown above.

We can see that if $k > 0$ is small, k will increase, so the equilibrium $k = 0$ is *unstable*. The graph of $k(t)$ will have an inflection point when k reaches $\left(\frac{s}{3\lambda}\right)^{3/2}$ (where dk/dt has its maximum). k will then converge asymptotically to the non-zero equilibrium.

The equilibrium $k = (s/\lambda)^{3/2}$ is *asymptotically stable*: any solution that starts near the equilibrium will converge to the equilibrium as $t \to \infty$. In fact, *all* solutions with $k(0) > 0$ will converge asymptotically to this equilibrium.

A *direction field* of a first order differential equation is created by taking a grid of (t, k) values, evaluating the right side of the differential equation at each point in the grid to find the slope of the solution $k(t)$ through that point, and then plotting a short line segment with that slope at the point. The following shows a direction field and some solutions of (9) when $s = 1$ and $\lambda = 1$.



What does this mean in terms of the capital K and the labor L? Since $k(t) = K(t)/L(t)$, and $L(t) = L_0 e^{\lambda t}$, if $k(t)$ converges to an asymptotically stable equilibrium k_1, then $K(t)$ must behave asymptotically like $k_1 L(t)$. This means that, in the long term, $K(t)$ must grow exponentially, with the same exponent as $L(t)$. This model predicts that in the long term, capital will grow exponentially along with the labor. If, for example, the capital is too low, it will rapidly increase until it becomes approximately proportional to the labor, and then it will settle into a long term behavior in which capital remains proportional to the labor.