Math 312 Lecture 3 *(revised)*
Solving First Order Differential Equations: Separable and Linear Equations

Warren Weckesser
Department of Mathematics
Colgate University

24 January 2005

This lecture describes two techniques for solving certain first order differential equations. The general form for a first order differential equation is

$$y'(t) = f(t, y) \tag{1}$$

1. Separable Equations

If f can be "separated" into a quotient of a function of t and a function of y as

$$f(t, y) = \frac{h(t)}{g(y)}, \tag{2}$$

there is a chance that the solutions can be found analytically. The differential equation may be written

$$g(y(t))y'(t) = h(t) \tag{3}$$

We now integrate both sides:

$$\int_0^t g(y(s))y'(s)\,ds = \int_0^t h(s)\,ds \tag{4}$$

The right side is an integral of a known function. To deal with the left side, we first let $p(y) = \int_{y_0}^{y} g(z)dz$. Then we use the chain rule:

$$\frac{d}{dt}p(y(t)) = p'(y)y'(t) = g(y)y'(t). \tag{5}$$

Thus, on the left side of (4), we have

$$\int_0^t g(y)y'(s)\,ds = \int_0^t \left(\frac{d}{ds}p(y(s))\right) ds = p(y(t)) = \int_{y_0}^{y} g(z)\,dz \tag{6}$$

The net result can be summarized with the following formal procedure:

1. Treat the derivative $\frac{dy}{dt}$ as a fraction and rewrite the differential equation as

$$g(y)dy = h(t)dt. \tag{7}$$

2. Integrate with respect to y on the left and with respect to t on the right. The constants of integration can be combined into one constant on the right after integrating.

3. Solve for y.

The *real* short summary is:

Separate, *Integrate*, *Isolate* (i.e. solve for y).

Note that even if f can be separated into $h(t)/g(y)$, there are two potential obstacles to this method. First, it might not be possible to evaluate the integrals. Second, it might not be possible to solve for y after integrating. In this case, we have an implicit equation relating y and t, which can still be useful in some cases.

Note that all autonomous first order differential equations are separable.

Example 1. We'll apply the method to

$$\frac{dp}{dt} = rp \tag{8}$$

In this case, separating gives

$$\frac{dp}{p} = rdt, \tag{9}$$

but note that we have assumed that $p \neq 0$. Integrating gives

$$\ln|p| = rt + C_0 \tag{10}$$

Exponentiate both sides to obtain

$$|p| = e^{rt+C_0} = C_1 e^{rt}, \quad \text{where} \quad C_1 = e^{C_0}. \tag{11}$$

Note that $C_1 > 0$. For $p > 0$, we have $|p| = p$, so $p = C_1 e^{rt}$. If $p < 0$, $|p| = -p$, so $p = -C_1 e^{rt}$. This is the same as saying the constant in front of e^{rt} is negative. Also note that $p(t) = 0$ is an equilibrium solution. We can combine the three cases $p > 0$, $p = 0$ and $p < 0$ into one solution

$$p(t) = Ce^{rt}, \tag{12}$$

where C is an arbitrary constant. To satisfy an initial condition $p(0) = P_0$, we let $C = P_0$.

Example 2. Now consider a population model in which the per capita growth rate varies periodically. This might be because of seasonal variation. The differential equation is

$$\frac{dp}{dt} = (r + a\cos(\omega t))p \tag{13}$$

Separating gives

$$\frac{dp}{p} = (r + a\cos(\omega t))\,dt \tag{14}$$

and integrating gives

$$\ln|p| = rt + \frac{a}{\omega}\sin(\omega t) + C_0 \tag{15}$$

From here, the work is similar to the previous example. The final concise formula for the solution is

$$p(t) = Ce^{rt+(a/\omega)\sin(\omega t)} \tag{16}$$

where C is an arbitrary constant.

2. Linear Equations

If the first order differential equation has the form

$$y'(t) = p(t)y + g(t), \tag{17}$$

it is called a *linear* equation. We can always express the solution to such an equation in terms of integrals. The only obstacle will be evaluating the integrals.

To derive the solution, we first simply move $p(t)y$ to the left:

$$y'(t) - p(t)y = g(t), \tag{18}$$

The left side looks vaguely like the result of applying the product rule of differentiation to a product of y and something else. If $\mu(t)$ is some function, then $(\mu y)' = \mu y' + \mu' y$, which suggests that we multiply (18) by μ (whatever it is) to obtain

$$\mu y'(t) - \mu p(t)y = \mu g(t), \tag{19}$$

and then determine what $\mu(t)$ should be by solving $\mu' = -\mu p(t)$. This is a separable equation; a solution is

$$\mu(t) = e^{-\int p(t)\,dt}. \tag{20}$$

We could include an arbitrary multiple of this function by multiplying it by an arbitraryconstant C, but we don't need it. Equivalently, when we find the integral $\int p(t)\,dt$, we may choose the constant of integration to be zero.

With this choice of μ, (19) becomes

$$(\mu y)' = \mu g(t). \tag{21}$$

Integrating both sides gives[1]

$$\mu y = \int \mu g(t)\,dt + C \tag{22}$$

and so we have the following solution to (17):

$$y = \frac{1}{\mu}\left(\int \mu g(t)\,dt + C\right) \quad \text{where} \quad \mu(t) = e^{-\int p(t)\,dt}. \tag{23}$$

Example 1. Consider the initial value problem

$$y' = 2y + 3, \quad y(0) = 5. \tag{24}$$

This problem is linear, with $p(t) = 2$ and $g(t) = 3$. (It is also separable, so there is more than one way to solve this problem.) The integrating factor is

$$\mu(t) = e^{-\int p(t)\,dt} = e^{-\int 2\,dt} = e^{-2t}, \tag{25}$$

[1] Note that I have explicitly included the integration constant C on the right. Normally, when we write an indefinite integral $\int q(t)\,dt$, the constant of integration is implicitly assumed to be part of the result. For example, $\int 2x\,dx = x^2 + C$. However, it is a common mistake to *forget* the constant, so I choose to include it explicitly.

and the solution is

$$\begin{aligned} y(t) &= e^{2t}\left(\int \left(e^{-2t}\right)(3)\ dt + C\right) \\ &= e^{2t}\left(-\frac{3}{2}e^{-2t} + C\right) \\ &= -\frac{3}{2} + Ce^{2t}. \end{aligned} \tag{26}$$

To satisfy the initial condition $y(0) = 5$, we have

$$y(0) = -\frac{3}{2} + C = 5 \implies C = \frac{13}{2}. \tag{27}$$

So the solution to the initial value problem is

$$y(t) = -\frac{3}{2} + \frac{13}{2}e^{2t}. \tag{28}$$

You should check the answer by substituting it back into (24).

Example 2.

$$y' = -\frac{y}{t} + t, \quad y(1) = 2. \tag{29}$$

In this case, $p(t) = -1/t$ and $g(t) = t$. Note that the initial condition is given at $t = 1$ rather than the usual $t = 0$. Nothing is wrong or difficult about that. We just have to remember to evaluate the solution at $t = 1$ when we solve for the constant to make the solution satisfy the initial condition. Also, since we can not divide by zero, we will assume that we are only interested in the solution where $t > 0$.

The integrating factor is

$$\mu(t) = e^{-\int p(t)\,dt} = e^{\int \frac{1}{t}\,dt} = e^{\ln|t|} = e^{\ln t} = t, \tag{30}$$

where, because we assume $t > 0$, we used $|t| = t$. The solution is

$$\begin{aligned} y(t) &= \frac{1}{\mu}\left(\int \mu g(t)\,dt + C\right) \\ &= \frac{1}{t}\left(\int t^2\,dt + C\right) \\ &= \frac{1}{t}\left(\frac{t^3}{3} + C\right) \\ &= \frac{t^2}{3} + \frac{C}{t} \end{aligned} \tag{31}$$

To satisfy the initial condition $y(1) = 2$, we have

$$y(1) = \frac{1}{3} + C = 2 \implies C = \frac{5}{3}. \tag{32}$$

Thus the solution to the initial value problem is

$$y(t) = \frac{t^2}{3} + \frac{5}{3t}. \tag{33}$$

Example 3. Consider the differential equation

$$y' = 2ty + 3. \tag{34}$$

The differential equation is linear, with p(t) = 2t and g(t)=3. The integrating factor is

$$\mu(t) = e^{-\int p(t)\,dt} = e^{-\int 2t\,dt} = e^{-t^2}, \tag{35}$$

and the solution is

$$\begin{aligned} y(t) &= e^{t^2}\left(\int \left(e^{-t^2}\right)(3)\ dt + C\right) \\ &= e^{t^2}\left(3\int e^{-t^2}\ dt + C\right) \end{aligned} \tag{36}$$

There is no analytical expression for the integral $\int e^{-t^2}\,dt$, so this is the best we can do.[2]

[2]Because the integral of e^{-t^2} appears frequently in mathematics, it has been given a name. You may have seen the *error function* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\,ds$.