# Math 312 Supplemental Notes <br> Finding Eigenvalues and Eigenvectors of $2 \times 2$ Matrices 

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In Linear Algebra, you learned how to find the eigenvalues and eigenvectors of matrices. In these notes, I'll point out some shortcuts for $2 \times 2$ matrices.

Let

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

To find the eigenvalues of $A$, we must solve $\operatorname{det}(A-\lambda I)=0$ for $\lambda$. (The expression $\operatorname{det}(A-\lambda I)$ is called the characteristic polynomial.) We have

$$
\begin{aligned}
\operatorname{det}(A-\lambda I) & =(a-\lambda)(d-\lambda)-b c \\
& =\lambda^{2}-(a+d) \lambda+(a d-b c) \\
& =\lambda^{2}-\operatorname{Tr}(\mathrm{A}) \lambda+\operatorname{det}(A)
\end{aligned}
$$

where $\operatorname{Tr}(\mathrm{A})=a+d$ is the trace of $A$. (The trace of a square matrix is the sum of the diagonal elements.) Then the eigenvalues are found by using the quadratic formula, as usual.

Now consider the problem of finding the eigenvectors for the eigenvalues $\lambda_{1}$ and $\lambda_{2}$. An eigenvector associated with $\lambda_{1}$ is a nontrivial solution $\overrightarrow{\mathbf{v}}_{1}$ to

$$
\begin{equation*}
\left(A-\lambda_{1} I\right) \overrightarrow{\mathbf{v}}=\overrightarrow{\mathbf{0}} . \tag{1}
\end{equation*}
$$

Now

$$
A-\lambda_{1} I=\left[\begin{array}{cc}
a-\lambda_{1} & b \\
c & d-\lambda_{1}
\end{array}\right]
$$

The matrix $A-\lambda_{1} I$ must be singular. That is precisely what makes $\lambda_{1}$ an eigenvalue. If a $2 \times 2$ matrix is singular, the second row must be a multiple of the first row (unless the first row is zero). Therefore, we know that putting $A-\lambda_{1} I$ into row echelon form must result in a row of zeros. Since we know this must be the case, there is no need to actually do it! All we need to find an eigenvector is the first row. In particular, if $\overrightarrow{\mathbf{v}}=\left[v_{1}, v_{2}\right]^{\top}$, then (1) implies

$$
\begin{equation*}
\left(a-\lambda_{1}\right) v_{1}+b v_{2}=0 \tag{2}
\end{equation*}
$$

We could solve this for, say, $v_{2}$ in terms of $v_{1}$, and give all the possible eigenvectors in terms of the arbitrary parameter $v_{1}$. (This is the eigenspace associated with the eigenvalue $\lambda_{1}$.) However, for the purpose of solving a system of differential equations, all we need is one eigenvector. An easy solution to (2) is $v_{1}=-b$ and $v_{2}=\left(a-\lambda_{1}\right)$. Thus (unless ( $a-\lambda_{1}$ ) and $b$ both happen to be zero), once we write down the matrix $A-\lambda_{1} I$, we can immediately get the eigenvector

$$
\overrightarrow{\mathbf{v}}_{1}=\left[\begin{array}{c}
-b \\
a-\lambda_{1}
\end{array}\right]
$$

If both $\left(a-\lambda_{1}\right)$ and $b$ are zero, we can use the second row to find an eigenvector:

$$
\overrightarrow{\mathbf{v}}_{1}=\left[\begin{array}{c}
d-\lambda_{1} \\
-c
\end{array}\right] .
$$

So, once we have an eigenvalue of a $2 \times 2$ matrix, it is very easy to find a corresponding eigenvector. This works even when the eigenvalue is complex. It will give a correct complex eigenvector.

## Example 1.

$$
A=\left[\begin{array}{cc}
1 & 2 \\
3 & -4
\end{array}\right]
$$

The characteristic polynomial is

$$
\lambda^{2}-(1+(-4)) \lambda+((1)(-4)-(2)(3))=\lambda^{2}+3 \lambda-10,
$$

so we find

$$
\lambda=\frac{-3 \pm \sqrt{9-4(-10)}}{2}=-5,2 .
$$

Let $\lambda_{1}=-5$ and $\lambda_{2}=2$. Now we'll find an eigenvector for each eigenvalue.
$\underline{\lambda_{1}=-5}$

$$
A-\lambda_{1} I=\left[\begin{array}{ll}
6 & 2 \\
3 & 1
\end{array}\right]
$$

As expected, we see that the second row is a multiple of the first. Using the shortcut discussed above, we can immediately find one eigenvector to be

$$
\overrightarrow{\mathbf{v}}_{1}=\left[\begin{array}{c}
-2 \\
6
\end{array}\right]
$$

Of course, since any nonzero multiple of an eigenvector is also an eigenvector, we could also choose

$$
\overrightarrow{\mathbf{v}}_{1}=\left[\begin{array}{c}
-1 \\
3
\end{array}\right]
$$

$\underline{\lambda_{2}=2}$

$$
A-\lambda_{2} I=\left[\begin{array}{cc}
-1 & 2 \\
3 & -6
\end{array}\right]
$$

In this case, a possible eigenvector is

$$
\overrightarrow{\mathbf{v}}_{2}=\left[\begin{array}{l}
-2 \\
-1
\end{array}\right]
$$

or, if we want to minimize the number of minus signs,

$$
\overrightarrow{\mathbf{v}}_{2}=\left[\begin{array}{l}
2 \\
1
\end{array}\right]
$$

## Example 2.

$$
A=\left[\begin{array}{cc}
-1 & -3 \\
4 & 3
\end{array}\right]
$$

The charateristic polynomial is

$$
\lambda^{2}-2 \lambda+9,
$$

and the eigenvalues are

$$
\lambda=\frac{2 \pm \sqrt{4-36}}{2}=1 \pm 2 \sqrt{-2}=1 \pm 2 \sqrt{2} i
$$

Let $\lambda_{1}=1+2 \sqrt{2} i$, and $\lambda_{2}=\lambda_{1}^{*}$. We'll find an eigenvector associated with the eigenvalue $\lambda_{1}$.
We have

$$
A-\lambda_{1} I=\left[\begin{array}{cc}
-1-(1+2 \sqrt{2} i) & -3 \\
4 & 3-(1+2 \sqrt{2} i)
\end{array}\right]=\left[\begin{array}{cc}
-2-2 \sqrt{2} i & -3 \\
4 & 2-2 \sqrt{2} i
\end{array}\right]
$$

By using the shortcut discussed above, we can immediately write down the eigenvector

$$
\overrightarrow{\mathbf{v}}_{1}=\left[\begin{array}{c}
3 \\
-2-2 \sqrt{2} i
\end{array}\right]
$$

(If we were solving a system of differential equations, we would then want to express $\overrightarrow{\mathbf{v}}_{1}$ as

$$
\overrightarrow{\mathbf{v}}_{1}=\left[\begin{array}{c}
3 \\
-2
\end{array}\right]+i\left[\begin{array}{c}
0 \\
-2 \sqrt{2}
\end{array}\right]
$$

so $\overrightarrow{\mathbf{a}}=\left[\begin{array}{c}3 \\ -2\end{array}\right]$ and $\overrightarrow{\mathbf{b}}=\left[\begin{array}{c}0 \\ -2 \sqrt{2}\end{array}\right]$.)

