Math 312 - Applied Math: Social Sciences **Spring 2005**

Homework Assignment 3

Due *Friday, February 25*

1. In this problem we look at solutions to the "Romeo and Juliet" model.

   Let's suppose that the constants in the problem are such that

$$\begin{aligned} \frac{dR}{dt} &= -J \\ \frac{dJ}{dt} &= 2R \end{aligned} \tag{1}$$

   This means that Juliet reacts twice as strongly, and in the opposite direction, to Romeo's love for her as Romeo does to Juliet's love for him. In vector and matrix form, this equation is

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \vec{\mathbf{x}}, \quad \text{where} \quad \vec{\mathbf{x}} = \begin{bmatrix} R \\ J \end{bmatrix} \tag{2}$$

   (a) Find the general solution.

   (b) Solve the initial value problem $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   (c) Sketch the phase portrait for this system.
   Check your answer by using the PPLANE Java applet. See the link at
   `http://math.colgate.edu/~wweckesser/math312/`

   (d) Classify the equilibrium $\vec{\mathbf{0}}$ as either a saddle, source, sink, spiral source, spiral sink, or center.

   (e) Determine the stability of the equilibrium $\vec{\mathbf{x}}$. (That is, state whether it is asymptotically stable, stable or unstable.)

2. Now consider the modification of the Romeo and Juliet model in which the emotional states of Romeo and Juliet diminish with time if they are not interacting with each other. In the model of Problem 1, if Romeo suddenly disappeared, Juliet's emotional state would be governed by $dJ/dt = 0$, which means she would be stuck in whatever state she happened to be in when Romeo vanished. A more realistic model would have Juliet's love for Romeo decay to zero if Romeo is not around. (Rinaldi calls this the *forgetting process* in his paper about Laura and Petrarch.) This suggests adding the term $-\alpha J$ to the formula for dJ/dt, where α is a positive constant. We'll also assume that, in the absence of Juliet, Romeo's love for Juliet also decays. For simplicity, let's assume that Romeo's love fades twice as fast as Juliet's, so the term we add to dR/dt is $-2\alpha R$. The revised model is

$$\begin{aligned} \frac{dR}{dt} &= -2\alpha R - J \\ \frac{dJ}{dt} &= 2R - \alpha J \end{aligned} \tag{3}$$

   (a) Find the general solution to this linear system of differential equations, assuming $\alpha > 0$. Be careful! The form of the solution depends on how big α is. Be sure to account for this in your answer.

(b) Classify the equilibrium $(0,0)$, and determine its stability. (As in part (a), your answer will depend on α.)

(c) Sketch the phase portrait for this system when $\alpha = 2$.

(d) Sketch the phase portrait when $\alpha = 4$.

3. We consider a further modification of the Romeo and Juliet model.

You probably found that when $\alpha > 0$, both $R(t)$ and $J(t)$ approach zero as $t \to \infty$. The model appears to say that in the long run, Romeo and Juliet will be indifferent towards each other–not a very romantic outcome!

Suppose we borrow an idea from the "Laura and Petrarch" model, and modify (3) to include constant terms a_R and a_J that account for the inherent "appeal" of Romeo and Juliet, respectively. Unlike α, these parameters are not necessarily positive.

(a) Write down the differential equations in this case.

(b) Find the new equilibrium solution (R_0, J_0). (R_0 and J_0 will depend on a_R and a_J.)

(c) Let $U = R - R_0$ and $V = J - J_0$. Show that when expressed in terms of U and V, the differential equations become

$$\frac{d\vec{\mathbf{y}}}{dt} = A\vec{\mathbf{y}}, \quad \text{where} \quad \vec{\mathbf{y}} = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} R - R_0 \\ J - J_0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -2\alpha & -1 \\ 2 & -\alpha \end{bmatrix}$$

(This means that adding the constant terms to the equation simply shifts the "origin" to a new location. This is a general result *for linear systems.*)

(d) Since the equation for $\vec{\mathbf{y}}(t)$ is the same as equation (3), we know that if $\alpha > 0$, $\vec{\mathbf{y}}(t)$ will approach $\vec{\mathbf{0}}$ asymptotically, which implies $R(t) \to R_0$ and $J(t) \to J_0$.

Find all the possible values of a_R and a_J for which Romeo and Juliet will approach a state of mutual attraction (i.e. for which $R(t)$ and $J(t)$ will *both* approach positive values as $t \to \infty$). Show this region by shading it in the (a_R, a_J) plane.

4. *Nondimensionalizing the SIR and SIQR equations.* The SIR model of disease spreading is

$$\begin{aligned} \frac{dS}{dt} &= -rSI \\ \frac{dI}{dt} &= rSI - \gamma I \\ \frac{dR}{dt} &= \gamma I \end{aligned} \tag{4}$$

A variation of the SIR model is the $SIQR$ model, in which some of the infectives are quarantined, so they no longer react with the susceptible (see Homework 2). The differential equations for the $SIQR$ model are

$$\begin{aligned} \frac{dS}{dt} &= -rSI \\ \frac{dI}{dt} &= rSI - \gamma I - \beta I \\ \frac{dQ}{dt} &= \beta I - \gamma Q \\ \frac{dR}{dt} &= \gamma I + \gamma Q \end{aligned} \tag{5}$$

Note that if we add the three equations of the SIR model, we obtain

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = \frac{d}{dt}(S + I + R) = 0,$$

so $S + I + R = N$, where N is the constant total population. Similarly, in the $SIQR$ model, $S + I + Q + R = N$ is constant.

(a) Show that by choosing the nondimensional variables $\tau = \gamma t$, $\mathcal{S} = S/N$, $\mathcal{I} = I/N$ and $\mathcal{R} = R/N$, the SIR model may be written

$$\begin{aligned}
\frac{d\mathcal{S}}{d\tau} &= -\rho\mathcal{S}\mathcal{I} \\
\frac{d\mathcal{I}}{d\tau} &= \rho\mathcal{S}\mathcal{I} - \mathcal{I} \\
\frac{d\mathcal{R}}{d\tau} &= \mathcal{I}
\end{aligned} \tag{6}$$

where $\rho = \frac{rN}{\gamma}$. With this choice of variables, we have $\mathcal{S} + \mathcal{I} + \mathcal{R} = 1$, so we can interpret $\mathcal{S}$, $\mathcal{I}$, and $\mathcal{R}$ as the fractions of the population that are susceptible, infective, and recovered (or "removed"), respectively.

(b) Show that a similar choice of nondimensional variables in the $SIQR$ model results in

$$\begin{aligned}
\frac{d\mathcal{S}}{d\tau} &= -\rho\mathcal{S}\mathcal{I} \\
\frac{d\mathcal{I}}{d\tau} &= \rho\mathcal{S}\mathcal{I} - \mathcal{I} - \sigma\mathcal{I} \\
\frac{d\mathcal{Q}}{d\tau} &= \sigma\mathcal{I} - \mathcal{Q} \\
\frac{d\mathcal{R}}{d\tau} &= \mathcal{I} + \mathcal{Q}
\end{aligned} \tag{7}$$

where $\rho = \frac{rN}{\gamma}$ and $\sigma = \frac{\beta}{\gamma}$.

5. The purpose of this problem is to study the behaviors of the SIR and $SIQR$ models. Near the bottom of the Math 312 web page (`http://math.colgate.edu/~wweckesser/math312/`) you'll find the heading **Solvers for Specific Systems**. Included here are links to web pages that will allow you to enter the parameters and initial conditions for the equations, and then see plots of the solutions[1].

   The solvers use the dimensional equations. As a result of the previous problem, we know there is no loss of generality in assuming that $\gamma = 1$, and that $S(0) + I(0) + R(0) = 1$ in the SIR solver and $S(0) + I(0) + Q(0) + R(0) = 1$ in the $SIQR$ solver.

   (a) In the SIR Solver page, leave the default values and click on **Show Solution**[2]. The plot will show the values of S, I and R for $0 \leq t \leq 20$. (In the default initial conditions, $I(0) = 0.005$, which means 0.5% of the population is initially infected.)
   Describe the behavior of the solution. What happens to $S(t)$, $I(t)$, and $R(t)$? Which, if any, appear to approach zero? Does the entire population eventually get the disease?

[1] Since the SI variables of the SIR model are decoupled from R, the SIR model could also be studied with one of the phase plane Java applets, such as PPLANE.

[2] If you go back to the Solver page, change some parameters, and hit **Show Solution** again, you may have to hit the Reload button in your browser to see the new plot.

(b) Before experimenting with the $SIQR$ model, think about the "meaning" of the parameter β.

i. Describe in your own words (and in complete sentences) the phenomena in the real world that are modeled by the terms involving β in the differential equations.

ii. How might a government change the value of β? What obstacles might there be to making β very large?

(c) Bring up the $SIQR$ solver in your web browser. Using the default values for all the other parameters and initial conditions, vary β from 0 to 2 and observe the behavior of the solutions. (You might have to adjust the duration of the solution plotted in order to be sure of the values that the solutions approach as $t \to \infty$.)

i. What typically happens to $S(t)$, $I(t)$, $Q(t)$ and $R(t)$? Which ones appear to approach zero as $t \to \infty$? Does the behavior seem plausible?

ii. Is there a value of β that ensures that 50 percent of the population never becomes infected? What about 90 percent? (Give the values of β with an accurary of two significant digits; for example, $\beta = 0.52$, not $\beta = 0.5$.)

Include printed copies of the solutions to go along with your description of the behavior of the solution. (Use the **PDF Version** link below the plot to obtain a printable version of the plot.)