Math 312 - Applied Math: Social Sciences **Spring 2005**

Homework Assignment 4

****** with revised Problem 4 ******

Due ***Friday, April 8***

1. The *logistic map*

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

can be interpreted as a population model with discrete time. We will restrict our attention to $r > 0$, and to the x interval $0 \le x \le 1$. If x_n is very small, then $x_n^2 \approx 0$, so $x_{n+1} \approx rx_n$. This says that when the population is small, it grows exponentially if $r > 1$, and it decays exponentially if $0 < r < 1$. When x is not small, the behavior is more complicated. In this problem, we consider just a few possibilities. We will use both analysis and computer experiments to explore what can happen.

In the following, $f(x) = rx(1 - x)$.

(a) Show that if $0 < r < 1$, the only fixed point in $0 \le x \le 1$ is $x = 0$, and this fixed point is asymptotically stable (i.e. it is an attractor). (Hint: Use the first derivative at the fixed point.)

(b) Use cobwebbing to illustrate the iterations of the map when $r = 0.75$ with initial conditions $x_0 = 0.25$ and $x_0 = 0.75$.

Use Prof. Schult's Iteration Program at

`http://math.colgate.edu/math312/Spring1999/iterate.html`

This program uses p_k instead of x_n. The default formula is the logistic map, so you will only have to change the coefficient in the formula, and the starting point.

(c) Show that for $r > 1$, there are two fixed points $x = 0$ and $x = x^*$ in $0 \le x \le 1$. The second fixed point x^* will depend on r. Show that the fixed point at $x = 0$ is unstable (i.e. it is a repellor). Also determine the range of r for which the second fixed point x^* is asymptotically stable.

(d) Use cobwebbing to illustrate the iterations of the map when $r = 1.5$ with initial conditions $x_0 = 0.1$ and $x_0 = 0.9$. (See part (b).) What is x^*, and what is $f'(x^*)$?

(e) Repeat (d) with $r = 2.75$.

(f) Repeat (d) with $r = 3.25$. (You should find that $f'(x^*) < -1$.) What appears to be happening? Interpret what you see in terms of a population.

(g) Repeat (d) with $r = 3.9$. Does there appear to be any pattern in the iterations of x_n? (Try increasing the number of time periods plotted.) Interpret what you see in terms of a population.

(*Moral*: Even a very simple map can exhibit surprisingly complicated behavior.)

2. In class we discussed the Lanchester model of a battle. In this model, the rate of loss of one army is simply proportional to the size of the other army. This leads to the linear system

$$\begin{aligned} \frac{dx}{dt} &= -by \\ \frac{dy}{dt} &= -cx. \end{aligned} \tag{2}$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0. \tag{3}$$

   (a) (We showed in class that the origin is a saddle point. Here we consider a different technique that works nicely in this case.) Multiply the first differential equation by $2cx$, multiply the second equation by $-2by$, and then add the second to the first. Explain why the result implies that

$$cx^2 - by^2 = cx_0^2 - by_0^2. \tag{4}$$

   This shows that the trajectories in the phase plane are hyperbolas. If you know b, c, x_0 and y_0, you can easily determine where the hyperbola crosses the x or y axis, and thereby determine which army wins the battle, and how large the surviving army is.

   (b) Suppose $b = c$. Show that by changing the time variable to $\tau = bt$, the parameter b no longer appears in the equations. Thus if $b = c$, we have the system

$$\begin{aligned} \frac{dx}{d\tau} &= -y \\ \frac{dy}{d\tau} &= -x. \end{aligned} \tag{5}$$

3. In the Battle of Trafalgar (1805), the British fleet of 40 ships, under the command of Admiral Nelson, met the combined French and Spanish fleet of 46 ships. Assume that the ships in all the fleets have similar capabilities, so $b = c$ in the Lanchester model.

   (a) Use the Lanchester Model to predict the outcome of the battle, and the size of the surviving fleet, if a single battle between the two fleets takes place. Make a plot of the corresponding trajectory in the phase plane. (You can use the PPLANE Java applet.)

   (b) The battle did not actually happen as in the previous part. Instead, it took place in stages. Nelson sent 32 of his ships against 23 French and Spanish ships; the other 8 British ships battled the remaining 23 French and Spanish ships.

   Use the Lanchester model to predict the outcome of each of these battles. How many ships remain in each fleet? (Round fractions down, so "18.74" ships is really 18 ships.)

   (c) If the remaining fleets from the two battles meet in a third battle, what is the outcome predicted by the Lanchester model?

(*Moral:* "Divide and conquer" is more complicated than you might expect!)

4. Consider the following discrete-time "predator-prey" model:

$$\begin{aligned} x_{n+1} &= 2(1-x_n)x_n - \beta x_n y_n \\ y_{n+1} &= \frac{4}{5}y_n + 3\beta x_n y_n \end{aligned} \tag{6}$$

where $\beta \geq 0$. In this model, x_n is the "prey" population, and y_n is the "predator" population. If $y_n = 0$ or $\beta = 0$, x_n grows according to the logistic map. If $\beta > 0$, and $y_n > 0$, the term $-\beta x_n y_n$ causes a decrease in x_n. This models the loss due to the predator.

The predator relies on the prey as a food source. If $x_n = 0$ or $\beta = 0$, the predator population decays exponentially. The term $3\beta x_n y_n$ causes in increase in y_{n+1}. This models the benefit of the food supply provided by the prey to the predator.

(a) This map has three fixed points. The first two are $(0,0)$ and $(1/2,0)$. The third depends on β. Find the third fixed point, and determine the range of β for which the third fixed point is meaningful in a population model. (That is, what is the range of β for which both the x and y components of the third fixed point are positive?)

(b) Find the Jacobian matrix of the map.

(c) Classify and determine the stability of the fixed points $(0,0)$ and $(1/2,0)$. (For the fixed point $(1/2,0)$, the classification will depend on β.)

(d) Classify and determine the stability of the third fixed point if

   i. $\beta = 3/15 = 0.2$

   ii. $\beta = 6/15 = 0.4$

   Use the web page

   `http://math.colgate.edu/~wweckesser/solver/DiscretePredPrey.shtml`

   to create plots of the phase space and time series for a sample solution for each of these values of β.

(e) *Extra Credit.* Find the exact value of β at which the eigenvalues of the Jacobian at the third fixed point change from real to complex.