Math 312: Applied Mathematics: Social Sciences **Spring 2006**

First Order Differential Equations
Solved Problems

1. Newton's Law of Cooling says that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the ambient temperature of the air.

   Express this statement as a differential equation. Use $k > 0$ for the proportionality constant, and define *with complete sentences* any other variables or constants that you use in the equation.

2. Consider the first order autonomous differential equation

$$\frac{dy}{dt} = (y-2)(y^2+6y+8).$$

   (a) Find the equilibrium solutions, sketch the graph of the right side of the equation as a function of y, and use this graph to sketch the "phase line" (i.e. on the y axis, add arrows between the equilibria that indicate whether $y(t)$ is increasing or decreasing).

   (b) In one set of axes, sketch several solutions ($y(t)$ vs. t), including all equilibrium solutions. (Other than the equilibria, you do not have to find the solutions analytically.) Please be neat enough so that: curves that should not cross do not cross; curves that should be increasing functions of t are increasing; and curves that should be decreasing functions of t are decreasing. Label the axes appropriately.

   (c) If $y(0) = 1$, determine $\lim_{t\to\infty} y(t)$ and $\lim_{t\to-\infty} y(t)$.

3. Solve the following initial value problem.

$$\frac{dy}{dt} = y^3, \quad y(0) = -2.$$

Solutions

1. Let t be time; let $h(t)$ be the temperature of the object at time t; let A be the ambient temperature; and let k be the proportionality constant in Newton's Law of Cooling. Then, since "*the rate of change of the temperature of the object*" is $\frac{dh}{dt}$, and "*the difference between the temperature of the object and the ambient temperature*" is $h - A$, the differential equation is

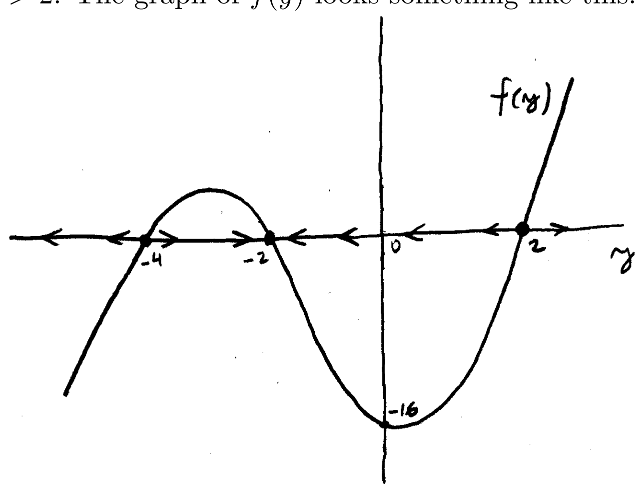
$$\frac{dh}{dt} = -k(h - A).$$

The minus sign is necessary for the equation to agree with our physical intuition: if $h > A$, the object is warmer than the ambient temperature, and it should cool off. This means $h(t)$ should decrease, so $\frac{dh}{dt}$ must be negative. We assumed $k > 0$, so the equation requires the minus sign.

2. (a) Let $f(y) = (y-2)(y^2+6y+8)$. To find the equilibrium solutions, we must solve $f(y) = 0$; that is,

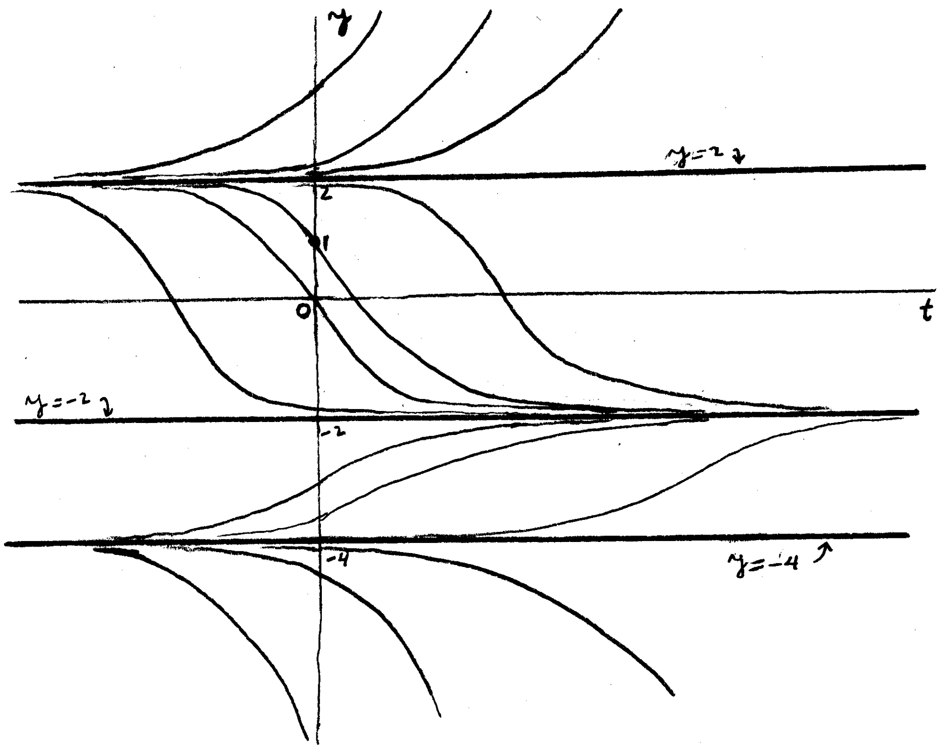
$$(y-2)(y^2+6y+8) = 0.$$

This polynomial factors as $(y-2)(y+2)(y+4)$, so the equilibrium solutions are $y(t) = 2$, $y(t) = -2$ and $y(t) = -4$. The polynomial is a cubic, and the coefficient of y^3 is 1, so we have: $f(y) < 0$ if $y < -4$; $f(y) > 0$ if $-4 < y < -2$; $f(y) < 0$ if $-2 < y < 2$; and $f(y) > 0$ if $y > 2$. The graph of $f(y)$ looks something like this:



The arrows on the y axis indicate the whether $y(t)$ is increasing or decreasing. (This is the "phase line" for the differential equation.)

(b) Here is a rough sketch of several solutions:



The horizontal lines $y = -4$, $y = -2$ and $y = 2$ are the equlibrium solutions. The other curves (not including the axes, of course) are rough sketches of additional solutions. Each solution has a different initial condition.

(c) If $y(0) = 1$, then $\lim_{t\to\infty} y(t) = -2$ and $\lim_{t\to-\infty} y(t) = 2$.
A rough sketch of this solution is included above.

3. This equation is separable. Separating gives

$$y^{-3}dy = dt,$$

and integrating gives

$$-y^{-2}/2 = t + c.$$

We use the initial condition $y(0) = -2$ to determine c. We have $c = -(-2)^{-2}/2 = -1/8$. Now solving for y gives

$$y = \frac{1}{\pm\sqrt{-2t + 1/4}}.$$

Because of the $\pm$, this is actually two solutions, and we must again use the initial condition to determine the sign. Since we want $y(0) = -2$, we must choose the negative sign. The solution to the initial value problem is

$$y = \frac{-1}{\sqrt{-2t + 1/4}}.$$

(Note that this solution is only defined for $t < 1/8$. The solution has a vertical asymptote at $t = 1/8$.)