B.3. Shortcuts for 2×2 Matrices

In this section, we give some shortcuts for finding the inverse of and the eigenvectors of 2×2 matrices.

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Inverse. You can easily check that the inverse is

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

So to find the inverse of a 2*x*2 matrix, *interchange the diagonal elements*, *change the sign of the off-diagonal elements*, and *divide by the determinant*.

EXAMPLE B.3.1.

$$A = \begin{bmatrix} 1 & 7 \\ -3 & 4 \end{bmatrix} \qquad A^{-1} = \frac{1}{25} \begin{bmatrix} 4 & -7 \\ 3 & 1 \end{bmatrix}$$

Eigenvalues and eigenvectors. To find the eigenvalues of A, we must solve $\det(A - \lambda I) = 0$ for λ. We have

$$\begin{aligned} \det(A - \lambda I) &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + (ad-bc) \\ &= \lambda^2 - \mathrm{Tr(A)}\lambda + \det(A) \end{aligned}$$

where $\mathrm{Tr(A)} = a + d$ is the *trace* of A. (The trace of a square matrix is the sum of the diagonal elements.) Then the eigenvalues are found by using the quadratic formula, as usual.

Now consider the problem of finding the eigenvectors for the eigenvalues λ_1 and λ_2. An eigenvector associated with λ_1 is a nontrivial solution $\vec{\mathbf{v}}_1$ to

$$(A - \lambda_1 I)\vec{\mathbf{v}} = \vec{\mathbf{0}}. \tag{B.4}$$

Now

$$A - \lambda_1 I = \begin{bmatrix} a - \lambda_1 & b \\ c & d - \lambda_1 \end{bmatrix}$$

The matrix $A - \lambda_1 I$ *must* be singular. That is precisely what makes λ_1 an eigenvalue. If a 2×2 matrix is singular, the second row *must* be a multiple of the first row (unless the first row is zero). Therefore, we know that putting $A - \lambda_1 I$ into row echelon form must result in a row of zeros. Since we know this must be the case, there is no need to actually do it! All we need to find an eigenvector is the first row. In particular, if $\vec{\mathbf{v}} = [v_1, v_2]^{\mathsf{T}}$, then (B.4) implies

$$(a - \lambda_1)v_1 + bv_2 = 0. \tag{B.5}$$

We could solve this for, say, v_2 in terms of v_1, and give all the possible eigenvectors in terms of the arbitrary parameter v_1. (This is the *eigenspace* associated with the eigenvalue λ_1.) However, often all we need is *one* eigenvector from this space. (More precisely, we want a *basis* for the eigenspace.) An easy solution to (B.5) is $v_1 = -b$ and $v_2 = (a - \lambda_1)$. Thus (unless $(a - \lambda_1)$ and b both happen to be zero), once we write down the matrix $A - \lambda_1 I$, we can immediately obtain the eigenvector

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} -b \\ a - \lambda_1 \end{bmatrix}$$

If both $(a-\lambda_1)$ and b are zero, we can use the second row to find an eigenvector:

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} d-\lambda_1 \\ -c \end{bmatrix}.$$

So, once we have an eigenvalue of a 2×2 matrix, it is very easy to find a corresponding eigenvector. This works even when the eigenvalue is complex. It will give a correct complex eigenvector.

EXAMPLE B.3.2.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^2 - (1+(-4))\lambda + ((1)(-4)-(2)(3)) = \lambda^2 + 3\lambda - 10,$$

so we find

$$\lambda = \frac{-3 \pm \sqrt{9-4(-10)}}{2} = -5, 2.$$

Let $\lambda_1 = -5$ and $\lambda_2 = 2$. Now we'll find an eigenvector for each eigenvalue.

$\underline{\lambda_1 = -5}$

$$A - \lambda_1 I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

As expected, we see that the second row is a multiple of the first. Using the shortcut discussed above, we can immediately find one eigenvector to be

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

Of course, since any nonzero multiple of an eigenvector is also an eigenvector, we could also choose

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$\underline{\lambda_2 = 2}$

$$A - \lambda_2 I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

In this case, a possible eigenvector is

$$\vec{\mathbf{v}}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

or, if we want to minimize the number of minus signs,

$$\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

EXAMPLE B.3.3.

$$A = \begin{bmatrix} -1 & -3 \\ 4 & 3 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^2 - 2\lambda + 9,$$

and the eigenvalues are

$$\lambda = \frac{2 \pm \sqrt{4-36}}{2} = 1 \pm 2\sqrt{-2} = 1 \pm 2\sqrt{2}\,i$$

Let $\lambda_1 = 1+2\sqrt{2}\,i$, and $\lambda_2 = \lambda_1^*$. We'll find an eigenvector associated with the eigenvalue λ_1.

We have

$$A - \lambda_1 I = \begin{bmatrix} -1-(1+2\sqrt{2}\,i) & -3 \\ 4 & 3-(1+2\sqrt{2}\,i) \end{bmatrix} = \begin{bmatrix} -2-2\sqrt{2}\,i & -3 \\ 4 & 2-2\sqrt{2}\,i \end{bmatrix}$$

By using the shortcut discussed above, we can immediately write down the eigenvector

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ -2-2\sqrt{2}\,i \end{bmatrix}$$

(If we were solving a system of differential equations, we would then want to express $\vec{\mathbf{v}}_1$ as

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -2\sqrt{2} \end{bmatrix}$$

so $\vec{\mathbf{a}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{\mathbf{b}} = \begin{bmatrix} 0 \\ -2\sqrt{2} \end{bmatrix}$.)