2.7. Second Order Differential Equations

Recall that the *order* of a differential equation is the order of the highest derivative in the equation. A general form of a second order differential equation is

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \tag{2.58}$$

It is an equation that relates a function to its first and second derivatives. For example,

$$\frac{d^2y}{dt^2} = -3\frac{dy}{dt} - 2y \tag{2.59}$$

says we want a function $y(t)$ with the property that its second derivative is equal to the given linear combination of the function and its first derivative for all t. You should verify that $y(t) = e^{-t}$ satisfies this equation, as does $y(t) = e^{-2t}$.

Second order differential equations often arise in models of mechanical systems, because Newton's second law of motion relates the *acceleration* of an object to the force applied to the object:

$$F = ma, \tag{2.60}$$

where F is the forced applied to the object, m is the mass of the object, and a is the acceleration of the object. For example, in the simplest model of a spring, the force exerted on a mass suspended on spring is proportional to the displacement of the object from its rest position. If we choose coordinates $y(t)$ so that $y = 0$ gives the rest position, the forced exerted on the object by the spring is $-ky$, where $k > 0$ is the spring constant. In this case, Newton's second law may be written

$$-ky = m\frac{d^2y}{dt^2}, \tag{2.61}$$

or, as it is more frequently written,

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0. \tag{2.62}$$

This is a second order differential equation for $y(t)$.

A trivial second order differential equation is

$$\frac{d^2y}{dt^2} = 0. \tag{2.63}$$

This one we can solve by simply integrating twice[4]:

$$\frac{dy}{dt} = A, \quad y = At + B, \tag{2.64}$$

where A and B are constants of integration. $y = At + B$ is a solution for any constants A and B. For a first order differential equation, we know we need an initial condition $y(0) = y_0$ to determine the value of the arbitrary constant that shows up in the solution. For a second order differential equation, there are generally two

[4]Note that we can not solve (2.59) by simply integrating. If we integrate once, we obtain $\frac{dy}{dt} = -3y - 2\int y(t)dt$. Since we don't know what $y(t)$ is (after all, $y(t)$ is what we are trying to find), we can not evaluate $\int y(t)\,dt$. Except for trivial cases (such as (2.63)), integrating both sides of a differential equation transforms it into a new problem, but does not solve it.

arbitrary constants, so we need *two* initial conditions: $y(0) = y_0$, $y'(0) = v_0$. So a general form for a second order initial value problem is

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \quad y(0) = y_0, \quad y'(0) = v_0. \tag{2.65}$$

(With these initial conditions, you should verify that the solution to the trivial differential equation $\frac{d^2y}{dt^2} = 0$ is $y(t) = v_0 t + x_0$.) In problems where $y(t)$ represents the position of an object, y_0 is the *initial position*, and v_0 is the *initial velocity*.

We can always convert an equation such as (2.58) into a system of two first order equations. Let $v(t) = \frac{dy}{dt}$; then $\frac{dv}{dt} = \frac{d^2y}{dt^2}$, and the single second order equation becomes the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= f(t, y, v) \end{aligned} \tag{2.66}$$

(This reinforces the idea that the initial value problem for a second order differential equations requires two initial conditions. We need starting values for both $y(t)$ and $v(t)$.)

EXAMPLE 2.7.1. Consider the second order differential equation

$$\frac{d^2y}{dy^2} + \mu(y^2 - 1)\frac{dy}{dt} + ky = 0. \tag{2.67}$$

We rewrite this equation as a system of two first order equations. Let $v = \frac{dy}{dt}$; then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -\mu(y^2 - 1)\frac{dy}{dt} - ky. \tag{2.68}$$

We replace $\frac{dy}{dt}$ with v and obtain the system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -\mu(y^2 - 1)v - ky \end{aligned} \tag{2.69}$$

Exercises

2.7.1. Convert the second order differential equation in (2.59) into a system of two first order equations.

2.8. Nondimensionalizing a Differential Equation

An important concept in mathematical modeling is that of *dimensionless* or *nondimensional* variables. In this section, we show how to rewrite a differential equation in terms of nondimensional variables and parameters. We will use an example to illustrate the procedure.

The projectile problem. We consider the problem of determining the height of an object that is launched vertically from the surface of the earth with initial speed v_0. Let t be the time, measured from the instant that the object is launched, let $x(t)$ be the height of the object above the surface of the earth, let g be the gravitational acceleration, and let R be the radius of the earth. Newton's laws may be used to derive the following differential equation for $x(t)$:

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x+R)^2}, \quad x(0) = 0, \quad x'(0) = v_0. \tag{2.70}$$

If we were going to perform computations with this equation and compare the solutions to actual experiments, we would need to work with a consistent set of units. For example, we might measure time in seconds (sec), distance in meters (m), and mass in kilograms (kg). In this case, the units of g are m/sec^2. The quantities time, length and mass are *dimensions*. For our equation to make sense, we must measure all dimensions with consistent units. Note that the dimension of a variable is an inherent property of the variable, but the units are something we can choose. For example, x is a length, but we might choose meters, miles, or even furlongs for its units. In the following, we will want to indicate the dimensions of all the variables and parameters in a problem. We'll use the symbols $\mathcal{L}$ for length, $\mathcal{T}$ for time, and $\mathcal{M}$ for mass.

The idea is to measure our variables in "units" that are instrinsic to the problem. Units such as kilometers or miles are arbitrary. The following procedure will let us choose units that can simplify the problem. Specifically, this procedure usually reduces the number of parameters in the problem.

Procedure for Nondimensionalizing a Differential Equation.

(1) List all the variables and parameters along with their dimensions.
(2) For each variable, say x, form a product (or quotient) p of parameters that has the same dimensions as x, and define a new variable $y = x/p$. The new variable y is a "dimensionless" variable. Its numerical value is the same no matter what system of units is used.
(3) Rewrite the differential equation in terms of the new variables.
(4) In the new differential equation, group the parameters into nondimensional combinations, and define a new set of nondimensional parameters expressed as the nondimensional combinations of the original parameters. (This will typically result in fewer parameters.)

We'll apply this procedure to the projectile problem, but first we point out an important aspect of step 3. Time t is one of the variables in the problem (usually we use it as the independent variable), so in step 2 we will create a nondimensional version of this variable, say τ. Since the differential equation has derivatives with respect to t, and we want the new equation to be expressed in terms of τ, we will have to use the chain rule to convert from t to τ. Suppose, for example, we have the dimensional variables t and $x(t)$, and we define nondimensional variables $\tau = t/T$

Variable or Parameter	Meaning	Dimension
t	time since the launch of the object	$\mathcal{T}$
x	distance from the surface of the earth	$\mathcal{L}$
g	gravitational constant	$\mathcal{L}\mathcal{T}^{-2}$
R	radius of the earth	$\mathcal{L}$
v_0	initial velocity	$\mathcal{L}\mathcal{T}^{-1}$

TABLE 1. The list of variables and parameters for the projectile problem, along with their dimensions. $\mathcal{T}$ means *time* and $\mathcal{L}$ means *length*.

and $y = x/P$. How do we express $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in terms of τ and y? First, write $y = x/P$ a little more carefully as

$$y(\tau) = \frac{x(T\tau)}{P} = \frac{x(t)}{P} \tag{2.71}$$

or

$$x(t) = Py(t/T) = Py(\tau) \tag{2.72}$$

Now take the derivative with respect to t on both sides. We will have to use the chain rule on the right.

$$\frac{dx(t)}{dt} = P\frac{d}{dt}\left(y(\tau)\right) = P\frac{dy}{d\tau}\frac{d\tau}{dt} = \frac{P}{T}\frac{dy}{d\tau} \tag{2.73}$$

I included the t and τ arguments in the first few expressions to remind you of what the arguments of x and y are, but from here on, I will suppress the arguments. The last equality in (2.73) comes from $\frac{d\tau}{dt} = 1/T$, since $\tau = t/T$. We also find

$$\frac{d^2x}{dt^2} = \frac{P}{T}\frac{d^2y}{d\tau^2}\frac{d\tau}{dt} = \frac{P}{T^2}\frac{d^2y}{d\tau^2} \tag{2.74}$$

Higher derivatives can be found the same way.

Once we understand how this works, we can take advantage of a formal shortcut. To express $\frac{dx}{dt}$ or $\frac{d^2x}{dt^2}$ in terms of τ and y, where $t = T\tau$ and $x = Py$, simply substitute the variables in the expression for the derivative:

$$\frac{dx}{dt} = \frac{d(Py)}{d(T\tau)} = \frac{P}{T}\frac{dy}{d\tau} \tag{2.75}$$

and

$$\frac{d^2x}{dt^2} = \frac{d^2(Py)}{d(T\tau)^2} = \frac{P}{T^2}\frac{d^2y}{d\tau^2} \tag{2.76}$$

This formal procedure seems fishy at first, but it is really just a shortcut for the chain rule.

We'll now apply the nondimensionalization procedure to the projectile problem.

Step 1. Table 1 shows the result of step 1. (I've also included the meaning of each variable and parameter in the list.)

Step 2. The variable t has dimension $\mathcal{T}$, so we must find a combination of the parameters that also has dimension $\mathcal{T}$. We see that R/v_0 is one such combination.

We define

$$\tau = \frac{t}{(R/v_0)} = \frac{v_0 t}{R} \tag{2.77}$$

The variable x has dimension $\mathcal{L}$, and so does R, so we define

$$y = \frac{x}{R} \tag{2.78}$$

Step 3. We now express (2.70) in terms of the dimensionless variables τ and y. We have $t = (R/v_0)\tau$ and $x = Ry$. Then, by using the shortcut discussed earlier, we have

$$\frac{dx}{dt} = \frac{d(Ry)}{d\left((R/v_0)\tau\right)} = v_0 \frac{dy}{d\tau} \tag{2.79}$$

and

$$\frac{d^2x}{dt^2} = \frac{d^2(Ry)}{d\left((R/v_0)\tau\right)^2} = \frac{R}{(R/v_0)^2}\frac{d^2y}{d\tau^2} = \frac{v_0^2}{R}\frac{d^2y}{d\tau^2} \tag{2.80}$$

Also note that when we substitute $x = Ry$ into the right side of the differential equation in (2.70), the R factors in the numerator and denominator cancel. To convert the initial conditions, we use (2.78) to obtain $y(0) = x(0)/R = 0$, and we use (2.79) to obtain $\frac{dy}{d\tau}(0) = \left(\frac{dx}{dt}(0)\right)/v_0 = 1$. The result of all this is the new equation

$$\frac{v_0^2}{R}\frac{d^2y}{d\tau^2} = -\frac{g}{(y+1)^2}, \quad y(0) = 0, \quad y'(0) = 1 \tag{2.81}$$

Step 4. Now we multiply both sides of the differential equation by R/v_0^2, and define β as

$$\beta = \frac{gR}{v_0^2} \tag{2.82}$$

Note that β is dimensionless. So our final, nondimensional problem is

$$\frac{d^2y}{d\tau^2} = -\frac{\beta}{(y+1)^2}, \quad y(0) = 0, \quad y'(0) = 1 \tag{2.83}$$

Instead of three parameters, we have just one, and everything is dimensionless. This is, in fact, a general result. When an equation is nondimensionalized, new parameters can be defined such that the equation depends only on the new parameters, which are all dimensionless.

The fact that the new variables and parameters are all dimensionless means that the equation does not change if we change our coordinates, say from miles and hours to meters and seconds. Also, the fact that we ended up with just one parameter means that the original three parameters (g, R, and v_0) did not have independent effects on the behavior. Any combinations of g, R and v_0 that result in the same value of β will result in the same behavior of the solution.

Let's go back and look at step 2 again. We defined $y = x/R$. This amounts to choosing the radius of the earth R as our fundamental unit of length. Given the nature of the problem, this is a "natural" or "intrinsic" length scale, as opposed to miles or meters, which are completely arbitrary.

We also defined $\tau = \frac{t}{R/v_0}$. Is there a natural or intrinsic meaning of R/v_0? You may recall that if an object moves at a constant velocity v_0, the distance that it travels in time T is $v_0 T$. On the other hand, if the object travels a distance R with constant velocity v_0, the time required is R/v_0. Thus, the "meaning" of R/v_0 is the time it would take an object to travel the radius of the earth if it were moving at

Variable or Parameter	Meaning	Dimension
t	time	$\mathcal{T}$
P	size of the population	$\mathcal{N}$
r	*per capita* growth rate of a small population	$\mathcal{T}^{-1}$
K	carrying capacity	$\mathcal{N}$
P_0	initial size of the population	$\mathcal{N}$

TABLE 2. The list of variables and parameters for the logistic equation, along with their dimensions. $\mathcal{T}$ means *time* and $\mathcal{N}$ means an *amount* or *quantity*.

the constant speed v_0. Unlike seconds or hours, which are arbitrary, R/v_0 provides a unit of time that is defined in terms of parameters in the problem; it is an intrinsic time scale.

Nondimensionalizing the Logistic Equation. Recall the logistic equation:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \quad P(0) = P_0, \tag{2.84}$$

where $r > 0$ and $K > 0$ are constants. We'll follow the steps outlined above to nondimensionalize this differential equation. Table 2 lists the variables and parameters.

To create the nondimensional time variable τ, we must divide t by something that has the dimension $\mathcal{T}$. The only choice here is $1/r$, so we define

$$\tau = \frac{t}{\left(\frac{1}{r}\right)} = rt. \tag{2.85}$$

To create the nondimensional dependent variable y, we must divide P by something that has the dimension $\mathcal{N}$. We have two choices here, K or P_0. I'll use K, and leave the choice of P_0 as an exercise. We have

$$y = \frac{P}{K} \tag{2.86}$$

For convenience, we also rewrite the definitions of τ and y as

$$t = \frac{\tau}{r}, \qquad P = Ky. \tag{2.87}$$

By the chain rule, we have

$$\frac{dP}{dt} = \frac{d(Ky)}{d(\tau/r)} = rK\frac{dy}{d\tau}. \tag{2.88}$$

Then substituting τ and y into (2.84) gives

$$rK\frac{dy}{d\tau} = r(1-y)Ky, \quad y(0) = \frac{P_0}{K}. \tag{2.89}$$

In the differential equation, the rK factors cancel, so the only parameters left are in the initial condition. Note that the fraction P_0/K is nondimensional. We define the new nondimensional initial condition

$$y_0 = \frac{P_0}{K} \tag{2.90}$$

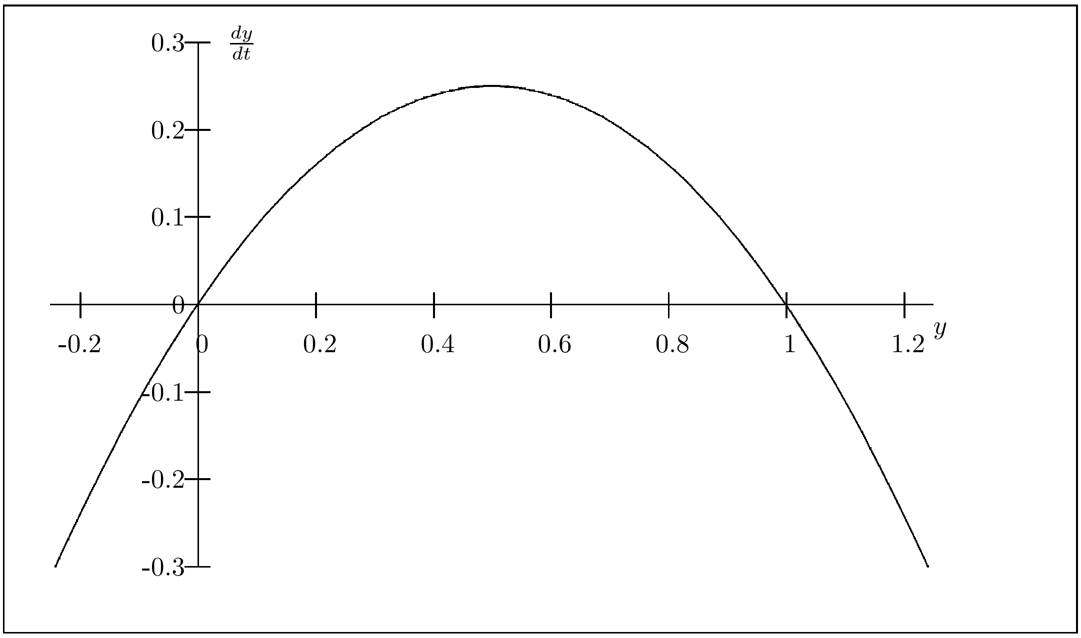

FIGURE 2.5. The plot of $\frac{dy}{dt}$ as a function of y for the nondimensional logistic equation (2.91).

to arrive at the nondimensional version of the logistic equation:

$$\frac{dy}{d\tau} = (1-y)y, \quad y(0) = y_0. \tag{2.91}$$

Now, instead of three parameters, we have just *one* nondimensional parameter. In a sense, all versions of the logistic equation (as written in (2.84)) behave the same; changing r or K simply amounts to changing the units of measurement.

Figure 2.5 shows the plot of $(1-y)y$, where we can see that there is a stable equilibrium at $y = 1$. Solutions to the nondimensional equation are shown in Figure 2.6.

How do we interpret the meaning of the nondimensional variables? We defined $y = P/K$, so this one is clear. K is a natural unit for the size of the population; y represents the population as a fraction of the carrying capacity. The carrying capacity determines an intrinsic unit of measurement for the population.

Can we find a similar interpretation for τ? We formed τ by dividing t by $1/r$ because the dimension of $1/r$ is time. Is there some intrinsic "meaning" to $1/r$? Consider a small population, where $P/K \ll 1$. In this case, the differential equation is approximately

$$\frac{dP}{dt} = rP,$$

and the solution is $P(t) = P_0e^{rt}$. Then $P(1/r) = P_0e$. Thus we can interpret $1/r$ as the time required for a small population to increase by a factor of e. (This unit of time is similar to the "half-life" of radioactive materials. The half-life is the time required for a given sample of the material to decay to half of the original amount.)

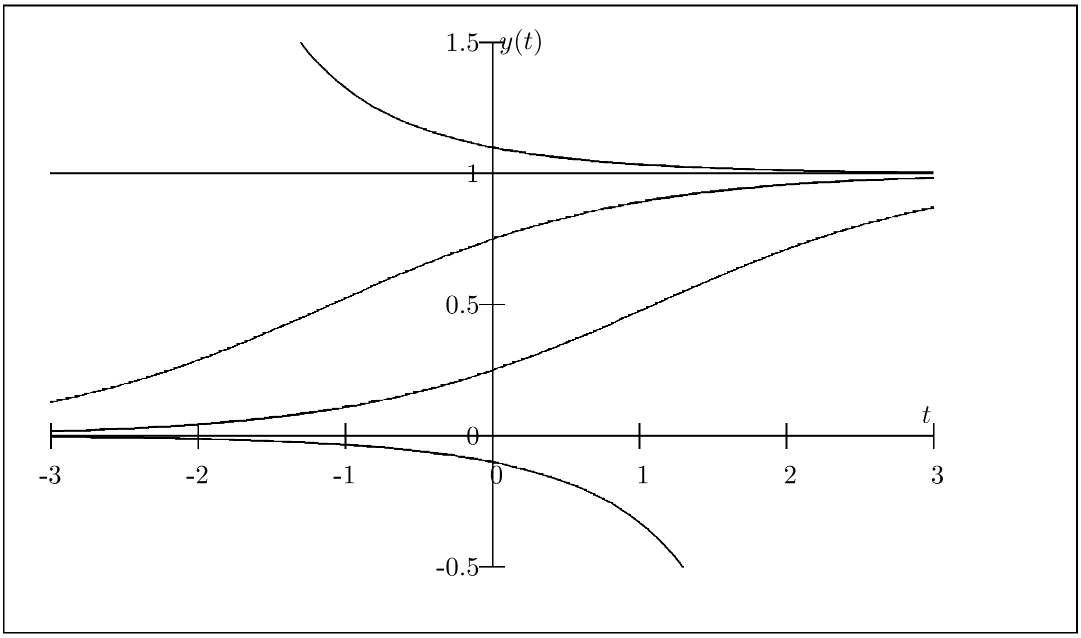

Figure 2.6. The plot of solutions to the nondimensional logistic equation (2.91) for several different initial conditions.

2.9. Dimensional Analysis and the Buckingham Pi Theorem

In this section we see how dimensional analysis can be used to discover properties of a system without solving, or even writing down, any differential equations.

2.9.1. Dimensional Homogeneity. We begin with the notion of *dimensional homogeneity*. An equation is *dimensionally homogeneous* if it is true regardless of the system of units that is used to measure the parameters or variable in the equation.

EXAMPLE 2.9.1. Consider the equation

$$s = \frac{gt^2}{2} \tag{2.92}$$

This is equation for the distance s that an object will fall when released at $t = 0$ in a constant gravitational field. If we use units of feet for distance and seconds for time, then $g = 32$ ft/sec^2. Suppose we convert to the units miles and hours for distance and time, respectively. We use a bar to indicate variables in the new units. We have $s = 5280\bar{s}$ (there are 5280 feet per mile), and $t = 3600\bar{t}$ (3600 seconds per hour). Finally we express g in the new units: since 1 ft = (1/5280) miles, and 1 sec = (1/3600) hr, we have $g = 32$ ft/sec^2 becomes $\bar{g} = 32(3600^2/5280)$ miles/hr^2. Let's substitute the new variables into (2.92):

$$\begin{aligned} 5280\bar{s} &= \frac{g\,(3600\bar{t})^2}{2} \\ \bar{s} &= \frac{3600^2}{5280}\frac{g\bar{t}^2}{2} \\ \bar{s} &= \frac{\bar{g}\bar{t}^2}{2} \end{aligned} \tag{2.93}$$

The new equation involving $\bar{s}$, $\bar{t}$ and $\bar{g}$ is the same as the original equation. This is an example of a dimensionally homogeneous equation. □

The power of dimensional analyis is based on the fundamental observation that *equations that arise from physical laws or real-world problems are dimensionally homogeneous*. The number of parameters in such an equation can generally be reduced, and this can lead to a better understanding of the system being studied.

2.9.2. The Period of a Pendulum. We consider a frictionless pendulum, as shown in Figure 2.7. Table 3 lists the parameters and their dimensions. (Note that angles, when expressed in radians, are actually dimensionless.) We consider an experiment in which we displace the pendulum by an angle θ_0, and release it with no initial velocity. Since we are ignoring friction, we expect the pendulum to oscillate. This oscillation will have some period T. (The period is the time required to complete one oscillation.) We would like to know how the period depends on the other parameters in the problem. First, we'll try to determine if all these parameters are really *independent*. To do this, we'll try to find all the nontrivial different ways that they can be combined to form dimensionless products. Our goal is to find the *dimensionless parameters*.

Consider the product

$$\pi = T^a l^b m^c g^d \theta_0^e \tag{2.94}$$

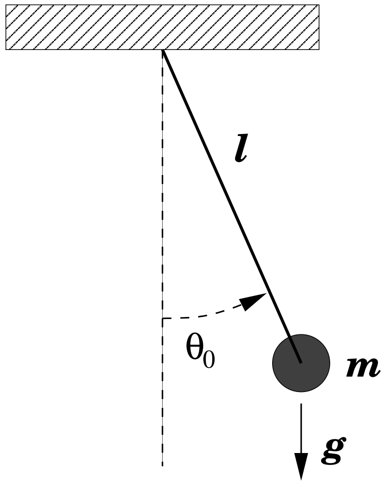

FIGURE 2.7. A pendulum of length l, mass m, acted on by gravity, released from the initial angle θ_0 with zero velocity.

Parameter	Meaning	Dimension
l	length of the pendulum	$\mathcal{L}$
m	mass of the pendulum bob	$\mathcal{M}$
g	gravitational acceleration	$\mathcal{L}\mathcal{T}^{-2}$
θ_0	initial angle	1
T	period of the oscillation	$\mathcal{T}$

TABLE 3. The list of variables and parameters for the pendulum, along with their dimensions. $\mathcal{L}$ means *length*, $\mathcal{M}$ means *mass*, and $\mathcal{T}$ means *time*. The initial angle θ_0 is dimensionless.

We want to choose a, b, c, d and e so that the new parameter π is dimensionless. (Note: π is the name of a parameter. We are not using $\pi = 3.1415\ldots$.) The dimensions of π are

$$\mathcal{T}^a\mathcal{L}^b\mathcal{M}^c\left(\mathcal{L}\mathcal{T}^{-2}\right)^d = \mathcal{T}^{a-2d}\mathcal{L}^{b+d}\mathcal{M}^c. \tag{2.95}$$

We want π to be dimensionless, so we want

$$\begin{aligned} a - 2d &= 0 \\ b + d &= 0 \\ c &= 0 \end{aligned} \tag{2.96}$$

This is a linear equation for the unknown a, b, c, and d. Actually, e is also an unknown, but it only shows up in the exponent of θ_0, and θ_0 is already dimensionless, so we know e is arbitrary. It is not difficult to solve the above system of equations, but I will still rewrite it in matrix form, and I'll include e in the system:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.97}$$

which we can write more concisely as

$$A\vec{\mathbf{p}} = \mathbf{0} \tag{2.98}$$

where A is the *dimension matrix* and $\vec{\mathbf{p}}$ is the vector of the powers a, b, c, d, e. Each nontrivial solution to the linear algebra problem provides a way to combine the dimensional parameters into a nondimensional product. Note, however, that if $\vec{\mathbf{p}} = [a, b, c, d, e]^{\mathsf{T}}$ is a solution, then so is $r\,[a, b, c, d, e]^{\mathsf{T}} = [ra, rb, rc, rd, re]^{\mathsf{T}}$ for any constant r. Since

$$T^{ra} l^{rb} m^{rc} g^{rd} \theta_0^{re} = \left(T^a l^b m^c g^d \theta_0^e\right)^r, \tag{2.99}$$

multiples of a solution to (2.97) do not really identify new combinations of parameters. Thus, all we need is a set of *linearly independent* solutions to (2.97). (To use the lingo from linear algebra, we need a *basis for the null-space of A*.) In this case, we see that the system (2.97) is already in reduced row echelon form, and the solution can be written

$$\vec{\mathbf{p}} = c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.100}$$

where c_1 and c_2 are arbitrary. Thus a basis for the null-space of A is given by the two vectors in the above solution. So one nondimensional parameter is

$$\pi_1 = T^2 l^{-1} m^0 g^1 \theta_0^0 = \frac{gT^2}{l} \tag{2.101}$$

and another (which we already knew) is

$$\pi_2 = T^0 l^0 m^0 g^0 \theta_0^1 = \theta_0. \tag{2.102}$$

Presumably there *is* some relationship among l, m, g, θ_0, and the period T. We don't know what it is, so we'll just assume it can be written in the form

$$f(T, l, m, g, \theta_0) = 0, \tag{2.103}$$

where f is dimensionally homogeneous. The fundamental result that we will now use is known as the *Buckingham Pi Theorem.* It says that a dimensionally homogeneous relation is equivalent to another relation expressed in terms of only the independent nondimensional parameters π_i. For our example, the Buckingham Pi Theorem implies that there is a function F for which

$$F(\pi_1, \pi_2) = 0. \tag{2.104}$$

Thus it must be possible to express the relation assumed in (2.103) in the simpler form

$$F\left(\frac{gT^2}{l}, \theta_0\right) = 0. \tag{2.105}$$

Equation (2.104) is an implicit relation between π_1 and π_2. We expect that for most values of π_1 and π_2, we can solve for π_1 in terms of π_2. That is, we can write (2.104) as

$$\pi_1 = h(\pi_2) \tag{2.106}$$

where h is some function. (In principle, the function h exists, but the dimensional analysis performed here tells us nothing about the nature of h.) Substituting the formulas for π_1 and π_2 into (2.106) gives

$$\frac{gT^2}{l} = h(\theta_0), \tag{2.107}$$

or

$$T = \sqrt{\frac{l}{g} h(\theta_0)} = \sqrt{\frac{l}{g}}\, \hat{h}(\theta_0) \tag{2.108}$$

where $\hat{h}(\theta_0) = \sqrt{h(\theta_0)}$. With this result, we can predict how the period of the oscillation of the pendulum depends on the parameters g and l, without actually solving (or even writing down) the differential equations that describe the motion.

For example, suppose a pendulum of length l_1 has period T_1 when released from angle θ_0. If the length is doubled and the pendulum is released from the same angle, the new period must be

$$T_2 = \sqrt{\frac{l_2}{g}}\, \hat{h}(\theta_0) = \sqrt{\frac{2l_1}{g}}\, \hat{h}(\theta_0) = \sqrt{2}\sqrt{\frac{l_1}{g}}\, \hat{h}(\theta_0) = \sqrt{2}\, T_1. \tag{2.109}$$

Thus, doubling the length should cause the period to increase by a factor of $\sqrt{2}$.

We can also compare the behavior of a pendulum on Earth to its behavior on Mars. The gravitational constant g_M on Mars is roughly one-third that of Earth's gravitational consant g_E. If, for a given initial angle θ_0 and length l, the period of the oscillation on Earth is 4 seconds, then on Mars the period will be

$$T_M = \sqrt{\frac{l}{g_M}}\, \hat{h}(\theta_0) = \sqrt{\frac{l}{g_E/3}}\, \hat{h}(\theta_0) = \sqrt{3}\sqrt{\frac{l}{g_E}}\, \hat{h}(\theta_0) = \sqrt{3}\, T_E = \sqrt{3}\, 4 \approx 6.9 \text{ seconds.} \tag{2.110}$$

2.9.3. Another Example: Periodic Drug Doses. We consider the concentration of a drug in the bloodstream of an individual when the drug is administered periodically. We'll look at a simple model that involves both a differential equation and a discrete map. We'll also see another example of using dimensional analysis.

Clearing of a Drug from the Bloodstream. There is a variety of mechanisms for a drug to be cleared from the bloodstream. Organs in the body may actively absorb the drug, the drug may react with other chemicals in the bloodstream, or the drug may naturally breakdown into simpler components. The simplest model of how a drug is cleared from the bloodstream results from assuming that the rate of loss of the drug is proportional to the concentration of the drug. If $c(t)$ is the concentration of the drug at time t, the differential equation for this simple model is

$$\frac{dc}{dt} = -rc, \tag{2.111}$$

where $r > 0$ is the proportionality constant that determines the rate of clearance. We know that the solution to this equation is $c(t) = c_0 e^{-rt}$, where c_0 is the concentration at time $t = 0$.

Equation (2.111) is the same as the equation for the decay of radioactive materials, so we can define the *half-life* of the drug. This is the time required for the amount of the drug to be reduced to half the original amount. For example, according to an article on **Slate** by Sam Schechner[5] about the poisoning of Ukrainian president Viktor Yushchenko, some isomers of dioxin have a half-life of more than seven years. If a drug has a half-life of seven years, then

$$c(7) = c_0 e^{-7r} = \frac{c_0}{2}, \tag{2.112}$$

which implies

$$r = \frac{\ln 2}{7} \approx 0.0990, \tag{2.113}$$

where we assumed t was measured in years, and therefore r has the units of years^{-1}.

QUESTION 2.9.1. *Suppose a drug has a half-life of seven years. (a) How long will it take for the amount of the drug to be reduced to one-quarter of the original amount? (b) How long will it take to be reduced to one percent of the original amount?*

Answers: *(a) 14 years, (b) 46.5 years.*

Administering a Drug with Periodic Doses. We now suppose that a drug is administered periodically. That is, every h time units, a dose is administered that increases the concentration by b. (If the *amount* administered is a, and the volume of the blood in bloodstream is V, then $b = a/V$. We will work with b from here on.) We assume that the dose causes an instantaneous increase in the concentration. This is probably a good assumption for a drug administered by an injection. Whether this is a good assumption for a drug taken orally depends on the properties of the drug. Alcohol, for example, enters the bloodstream fairly rapidly, so the instantaneous increase in concentration is probably a reasonable approximation for alcohol.

We assume that the first dose is administered at time $t = 0$, and the concentration of the drug in the bloodstream before then is zero. At $t = 0$ the concentration jumps to b, and then for $0 < t < h$, the concentration decays according to (2.111). At $t = h$, another dose is administered, and the concentration increases by b. The concentration then decays for $h < t < 2h$, and the process continues. We expect the plot of the concentration to look like the graph shown in Figure 2.8.

Let x_n be the concentration at the moment before a new dose is administered. It is clear the graph of $c(t)$ will consists of periods of decay, separated by jumps. What we would like to know is what happens to x_n as n increases? Does x_n increase without bound? Or does x_n approach some value asymptotically? If so, what value does is approach?

Dimensional Analysis of x_∞. We'll find the exact formula for x_n in the next section. In this section, we assume that x_n approaches a finite value x_∞ asymptotically as $n \to \infty$. We use dimensional analysis to determine (as far as possible) how x_∞ depends on the other parameters.

[5] `http://slate.msn.com/id/2110979/`, Dec. 13, 2004

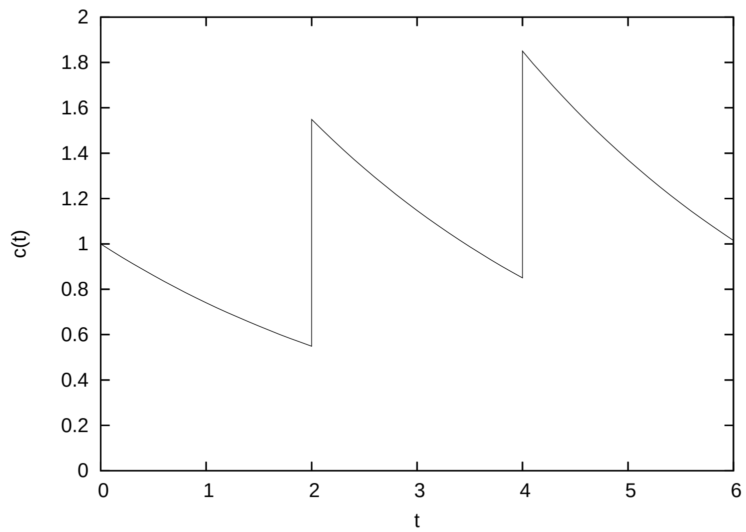

FIGURE 2.8. A plot of the concentration $c(t)$ for a drug administered periodically. In this example, $r = 0.3$, $h = 2$ and $b = 1$. At $t = 0, h, 2h, \ldots$, $c(t)$ increases by b; otherwise the concentration decays according to (2.111).

Parameter	Meaning	Dimension
r	proportionality constant for the clearance of the drug	$\mathcal{T}^{-1}$
h	time period	$\mathcal{T}$
b	instantaneous change in concentration due to each dose	$\mathcal{C}$
x_∞	asymptotic concentration just before the next dose	$\mathcal{C}$

TABLE 4. The list of variables and parameters along with their dimensions. $\mathcal{T}$ means *time* and $\mathcal{C}$ means a *concentration*. (In a problem with a wider variety of parameters, we might want to express concentration as an *amount* $\mathcal{A}$ divided by a *volume* $\mathcal{V}$ or even divided by the cube of a *length* $\mathcal{L}$, but this is not necessary in this case.)

We use the procedure that we saw for the period of the pendulum in an earlier lecture. Our first task is to find all the *independent* nondimensional parameters. We list all the relevant parameters along with their dimensions in Table 4. We see that $\pi_1 = rh$ and $\pi_2 = x_\infty/b$ are nondimensional combinations of the parameters. With a little thought, we could probably convince ourselves that these are the *only* nontrivial (or independent) combinations. (For example, rhb^2/x_∞^2 is nondimensional, but it is equivalent to π_1/π_2^2, so it is not really a "new" parameter.) However, for the sake of pedagogy, we will follow the formal procedure, and pretend we didn't see the "obvious" choices.

We must choose exponents α, β, γ and δ such that

$$\pi = r^\alpha b^\beta h^\gamma x_\infty^\delta \tag{2.114}$$

is dimensionless. Substituting in the dimensions from Table 4, we require

$$\left(\mathcal{T}^{-1}\right)^\alpha \mathcal{C}^\beta \mathcal{T}^\gamma \mathcal{C}^\delta = \mathcal{T}^0\mathcal{C}^0 = 1 \quad \Longrightarrow \quad \mathcal{T}^{-\alpha+\gamma}\mathcal{C}^{\beta+\delta} = \mathcal{T}^0\mathcal{C}^0. \tag{2.115}$$

This results in the linear system of equations

$$\begin{aligned} -\alpha + \gamma &= 0, \\ \beta + \delta &= 0. \end{aligned} \tag{2.116}$$

This is easy enough to solve: $\alpha = \gamma$ and $\beta = -\delta$, where γ and δ are arbitrary. Equivalently, we can express this as

$$\alpha = p, \quad \beta = -q, \quad \gamma = p, \quad \delta = q, \tag{2.117}$$

where p and q are arbitrary parameters. In vector form,

$$\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{bmatrix} = \begin{bmatrix}p\\ -q\\ p\\ q\end{bmatrix} = p\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + q\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}. \tag{2.118}$$

A *basis* for the solution set of (2.116) is given by the vectors $\{[1,0,1,0],[0,-1,0,1]\}$ (written as row vectors for convenience). Each basis vector gives us exponents that we can plug into (2.114) to form a nondimensional parameter. Thus we have found (as expected) that there are only two independent nondimensional parameters:

$$\pi_1 = r^1 b^0 h^1 x_\infty^0 = rh, \quad \text{and} \quad \pi_2 = r^0 b^{-1} h^0 x_\infty^1 = \frac{x_\infty}{b}. \tag{2.119}$$

Now we assume that there is some functional relationship among r, h, b and x_∞. Since we don't know what it is, we'll assume the general form

$$f(r,h,b,x_\infty) = 0. \tag{2.120}$$

We expect f to be dimensionally homogeneous; then the Buckingham Pi Theorem implies that there is an equivalent relationship of the form

$$F(\pi_1,\pi_2) = 0. \tag{2.121}$$

Moreover, we expect that for most values of π_1 and π_2, this equation can be solved for π_2 in terms of π_1. That is, there is some function G such that (2.121) is equivalent to

$$\pi_2 = G(\pi_1). \tag{2.122}$$

Subsituting in the definitions of π_1 and π_2 gives

$$\frac{x_\infty}{b} = G(rh), \tag{2.123}$$

or

$$x_\infty = bG(rh). \tag{2.124}$$

This gives us the form of the equation that will result if we can solve for x_∞ in terms of r, h and b. That is, x_∞ *must* be a product of b and some function of rh only.

This problem is actually simple enough that we can solve for x_∞ exactly. In the steady-state behavior of $c(t)$, the decrease in the concentration during the time interval between doses must be exactly b. For convenience, let us shift our time axis so that the concentration has just jumped to $c_{\max}$ at $t = 0$. Then $c(h) = c_{\max}e^{-rh}$, and the change in the concentration is

$$c(0) - c(h) = c_{\max} - c_{\max}e^{-rh} = c_{\max}(1 - e^{-rh}). \tag{2.125}$$

This must equal b:

$$c_{\max}(1 - e^{-rh}) = b \implies c_{\max} = \frac{b}{1 - e^{-rh}} \tag{2.126}$$

Finally, since x_∞ is the concentration at the end of the h time interval (just before the next dose), we have

$$x_\infty = c_{\max} - b = \frac{be^{-rh}}{1 - e^{-rh}}. \tag{2.127}$$

As expected, the formula has the form $bG(rh)$. In this case, $G(u) = \frac{e^{-u}}{1-e^{-u}}$.

Derivation of the Formula for x_n. In this section,[6] we derive the actual formula for x_n. You may find it helpful to label the plot in Figure 2.8 using the notation from the following discussion.

We need some additional notation to describe the graph shown in the figure. Let

$$\begin{aligned} c(h^-) &= \lim_{t \to h^-} c(t) \qquad \text{(the limit from below)}, \\ c(h^+) &= \lim_{t \to h^+} c(t) \qquad \text{(the limit from above)}. \end{aligned} \tag{2.128}$$

and define

$$x_n = c((nh)^-). \tag{2.129}$$

The instant after the first dose, we have

$$c(0^+) = b. \tag{2.130}$$

Then, for $0 < t < h$, we have $c(t) = be^{-rt}$, so

$$c(h^-) = be^{-rh}. \tag{2.131}$$

At $t = h$, the concentration increases by b, so

$$c(h^+) = c(h^-) + b = be^{-rh} + b = b\left(e^{-rh} + 1\right) \tag{2.132}$$

Then in the next interval, the solution again decays, and we have

$$c((2h)^-) = c(h^+)e^{-rh} = b\left(e^{-rh} + 1\right)e^{-rh} = b\left(e^{-2rh} + e^{-rh}\right) \tag{2.133}$$

and after the jump at $t = 2h$ we have

$$c((2h)^+) = c((2h)^-) + b = b\left(e^{-2rh} + e^{-rh}\right) + b = b\left(e^{-2rh} + e^{-rh} + 1\right) \tag{2.134}$$

Once again, in the next time interval, the solution decays and we have

$$c((3h)^-) = c((2h)^+)e^{-rh} = b\left(e^{-2rh} + e^{-rh} + 1\right)e^{-rh} = b\left(e^{-3rh} + e^{-2rh} + e^{-rh}\right) \tag{2.135}$$

Recall that we defined $x_n = c((nh)^-)$. The process that we are describing defines a one dimensional mapping

$$x_{n+1} = (x_n + b)e^{-rh}, \quad \text{with } x_0 = 0. \tag{2.136}$$

[6] This section is not an integral part of the discussion of dimensional analysis.

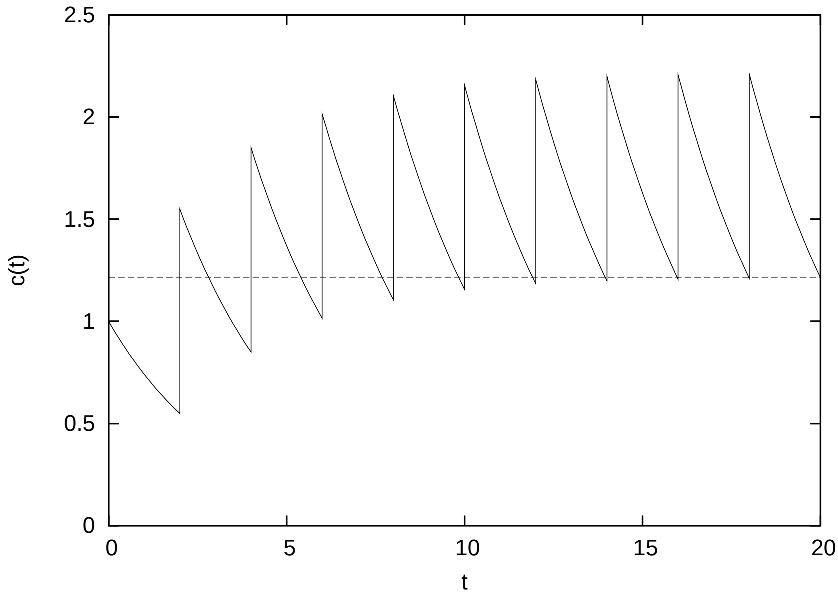

FIGURE 2.9. The graph of $c(t)$ for $r = 0.3$, $h = 2$, and $b = 1$. The dashed line shows $x_\infty \approx 1.2164$.

Equations (2.131), (2.133) and (2.135) give the formulas for the first three iterations of this map. In general, we have

$$x_n = c((nh)^-) = b\left(e^{-nrh} + e^{-(n-1)rh} + \cdots + e^{-rh}\right) = b\sum_{k=1}^{n} e^{-krh} = b\sum_{k=1}^{n}\rho^k, \tag{2.137}$$

where $\rho = e^{-rh}$. (Note that, since $r > 0$ and $h > 0$, we have $0 < \rho < 1$.) The formula in Equation (2.137) is a geometric sum. By using Equation (A.6) from the appendix, we obtain

$$x_n = b\rho\left(\frac{1-\rho^n}{1-\rho}\right) = be^{-rh}\left(\frac{1-e^{-nrh}}{1-e^{-rh}}\right) \tag{2.138}$$

Finally, since $\lim_{n\to\infty}\rho^n = 0$, we have

$$x_\infty = \frac{b\rho}{1-\rho} = \frac{be^{-rh}}{1-e^{-rh}}. \tag{2.139}$$

The example shown earlier was for $r = 0.3$, $h = 2$ and $b = 1$. With these values we find $x_\infty \approx 1.2164$. Figure 2.9 shows the result of ten periods for these parameters. The dashed line in this plot indicates x_∞.

Exercises

2.9.1. Find a set of independent nondimensional parameters for the following dimensional parameters. The dimension of each parameter is given in parentheses.

$$\ell\ (\mathcal{L}),\ r\ (\mathcal{T}^{-1}),\ g\ (\mathcal{L}\mathcal{T}^{-2}),\ m\ (\mathcal{M}),\ P\ (\mathcal{M}\mathcal{L}^{-1}\mathcal{T}^{-2})$$