3.4. Battle of Attrition–The Lanchester Model

We consider a model of a battle in which the opponents simply blast away at each other until one side is wiped out. This model was proposed by F. W. Lanchester[**5**]. The simple version presented here is *not* a realistic model of modern warfare! In this model $x(t)$ and $y(t)$ are the sizes of the opposing armies at time t. As the battle rages, the losses incurred by army x are simply proportional to the size of army y, and the losses incurred by army y are proportional to the size of army x (but not necessarily with the same proportionality constant). That is,

$$\begin{aligned}\frac{dx}{dt} &= -\alpha y\\ \frac{dy}{dt} &= -\beta x\end{aligned} \tag{3.34}$$

where α and β are positive constants. This is a planar linear system. The coefficient matrix is

$$A = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix}. \tag{3.35}$$

The eigenvalues are $\lambda_1 = -\sqrt{\alpha\beta}$ and $\lambda_2 = \sqrt{\alpha\beta}$, so we know that the origin $(0,0)$ is a saddle point. A corresponding set of eigenvectors are

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ -\sqrt{\frac{\beta}{\alpha}} \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ \sqrt{\frac{\beta}{\alpha}} \end{bmatrix}.$$

The x nullcline is $y = 0$, which means that trajectories cross the x axis vertically. Similarly, the y nullcline is $x = 0$, so trajectories cross the y axis horizontally. Since the nullclines are the axes, we know that in the first quadrant, $dx/dt < 0$ and $dy/dt < 0$, so for any initial condition in the first quadrant, the trajectory will eventually cross either the x or y axis. If the initial condition happens to be in the eigenspace associated with the negative eigenvalue λ_1, the trajectory will converge to $(0,0)$. (In this exceptional case, the armies wipe each other out!) This eigenspace is given by the line

$$y = \sqrt{\frac{\beta}{\alpha}}\,x. \tag{3.36}$$

If the initial condition is above the λ_1 eigenspace (so $y_0 > \sqrt{\frac{\beta}{\alpha}}\,x_0$), then y wins, and if the initial condition is below this line, then x wins.

There is a different approach we can take to analyze this system. By multiplying the first equation by $2\beta x$ and the second by $-2\alpha y$, we obtain

$$\begin{aligned}2\beta x\frac{dx}{dt} &= -2\alpha\beta xy\\ -2\alpha y\frac{dy}{dt} &= 2\alpha\beta xy.\end{aligned} \tag{3.37}$$

Then adding these equations results in

$$2\beta x\frac{dx}{dt} - 2\alpha y\frac{dy}{dt} = 0. \tag{3.38}$$

We note that the left side of this equation is the t derivative of $\beta x^2 - \alpha y^2$. So we'll integrate from 0 to t:

$$\int_0^t \left(2\beta x\frac{dx}{dt} - 2\alpha y\frac{dy}{dt}\right)ds = \int_0^t 0\,ds = 0. \tag{3.39}$$

which gives

$$\begin{aligned}
\beta x^2(s) - \alpha y^2(s)\big|_{s=0}^{s=t} &= 0 \\
\beta x^2(t) - \alpha y^2(t) - (\beta x^2(0) - \alpha y^2(0)) &= 0 \qquad (3.40)\\
\beta x^2(t) - \alpha y^2(t) &= \beta x^2(0) - \alpha y^2(0) = \beta x_0^2 - \alpha y_0^2
\end{aligned}$$

By dropping the explicit t dependence of $x(t)$ and $y(t)$, we obtain

$$\beta x^2 - \alpha y^2 = \beta x_0^2 - \alpha y_0^2. \qquad (3.41)$$

This is the equation of a hyperbola in the (x,y) plane. Whether the hyperbola crosses the x or y axis is determined by the sign of $\beta x_0^2 - \alpha y_0^2$. If this value is positive, then the hyperbola crosses the positive x axis at $x = \sqrt{x_0^2 - \frac{\alpha}{\beta} y_0^2}$. If $\beta x_0^2 - \alpha y_0^2$ is negative, then the hyperbola crosses the y axis at $y = \sqrt{y_0^2 - \frac{\beta}{\alpha} x_0^2}$. Thus, if we know the initial sizes of the armies and the constants α and β, we can easily determine which army will win, and what the size of the winning army will be at the end of the battle.

EXAMPLE 3.4.1. Suppose we have two armies, with $\alpha = 1.0$, $\beta = 1.5$, and $x_0 = 10,000$. What is the outcome of the battle if $y_0 = 10,000$? What if $y_0 = 14,000$? How large must y_0 be if both armies wipe each other out?

We can answer these questions with equation (3.41). We have

$$1.5x^2 - y^2 = 1.5(10000)^2 - y_0^2 \qquad (3.42)$$

If $y_0 = 10,000$, we have

$$1.5x^2 - y^2 = 1.5(10000)^2 - (10000)^2 = 5 \times 10^7. \qquad (3.43)$$

The right side is positive, so the hyperbola must cross the x axis. That is, the x army wins, and the size of the surviving army is

$$x = \sqrt{\frac{5 \times 10^7}{1.5}} \approx 5800. \qquad (3.44)$$

If $y_0 = 14,000$, we have

$$1.5x^2 - y^2 = 1.5(10000)^2 - (14000)^2 = -4.6 \times 10^7. \qquad (3.45)$$

The right side is negative, so the hyperbola must cross the y axis. The y army wins, and its remaining size is

$$y = \sqrt{4.6 \times 10^7} \approx 6780. \qquad (3.46)$$

The armies will wipe each other out when

$$y_0 = \sqrt{\frac{\beta}{\alpha}} x_0 = \sqrt{\frac{1.5}{1.0}} 10000 \approx 12247. \qquad (3.47)$$

Plots of the three curves in the xy plane for each of these cases are shown in Figure 3.3. □

Equation (3.36) gives an asymptote of the family of hyperbolas. If the initial sizes of the armies result in a point on this line, the two armies will eliminate each other. The equation lets us make an interesting observation. Suppose x_0 is, say, three times larger than y_0. How much more efficient must the y army be in order to defeat the larger opponent? In other words, by what factor must α exceed β to ensure that $y_0 > \sqrt{\frac{\beta}{\alpha}} x_0$? According to this equation, the y army must be more than *nine times* more effective than the x army if it hopes to win the battle. In this simple model, raw numbers are very important.

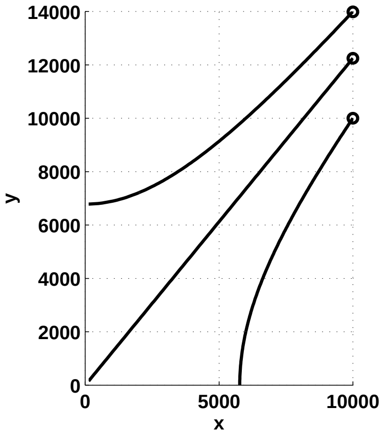

FIGURE 3.3. Plots of the three curves discussed in Example 3.4.1.

Exercises

3.4.1. A colony of $50,000$ red ants is in a battle with a colony of $35,000$ larger and stronger black ants. The sizes and strengths of the two types of ants are such that $\alpha = 3\beta$.

(a) Based on this description, which type of ant (black or red) is represented by x and which is represented by y in the Lanchester model?
(b) If the battle proceeds according to the Lanchester model, which colony will win, and how many ants will remain in the winning colony?

3.4.2. In the year 17**, a general is facing an impending battle. Across the valley from his encampment, the opposing army has $10,000$ soldiers, but the general has only $6,500$. His army and the opposing army have similar abilities and armaments, so, in a Lanchester model of the battle, the proportionality constant α and β are equal. If, as might be the case in these times, the battle is accurately modeled by the Lanchester equations, the general's army will lose the battle. Fortunately, the general has $5,000$ reinforcement on the way. Unfortunately, the opposing army also expects reinforcements, but only $2,000$. This is enough to give the advantage to the opposing army.

After consulting his maps, the general realizes that he can divert his reinforcements to engage the opposing army's reinforcements before they reach the valley. According to the Lanchester model, the general's reinforcements would defeat the opposing army's reinforcements. The surviving reinforcements could then join the general's army before the general engages in the battle with the opposing army across the valley.

(a) Show that, if the surviving reinforcements reach the valley before the main battle begins, the general will win the main battle. What will be the size of the surviving army?
(b) The general realizes that the opposing army might attack before his surviving reinforcements reach him. If the reinforcements arrive too late, his losses might have been too great, and the reinforcements will not be enough to win the battle. How large can the pre-reinforcment losses be before the reinforcement will not be enough to win the main battle?

3.4.1

(a) Since $\alpha = 3\beta$, $\alpha > \beta$. We are told that the black ants are larger and stronger than the red ants, so the size of the black colony must be y, and the size of the red colony is x.

(b) We have

$$\beta x^2 - 3\beta y^2 = \beta x_0^2 - 3\beta y_0^2.$$

We see that we can divide by β on both sides. Putting in the initial sizes of the colonies gives

$$x^2 - 3y^2 = (50000)^2 - 3(35000)^2 < 0.$$

The right side is negative, so the curve must cross the y axis. This means the black colony will win. When $x = 0$, we find

$$y = \sqrt{\frac{3(35000)^2 - (50000)^2}{3}} \approx 20000.$$

So there will be approximately $20,000$ ants remaining in the black colony.