2.12. Planar Linear Systems

We consider the linear system of two first order differential equations

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$
(2.147)

or equivalently,

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \text{where} \quad \vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
(2.148)

■ EXAMPLE 2.12.1 Before discussing a general technique for solving such systems, we consider this example:

$$\frac{dx}{dt} = -3x + y$$
$$\frac{dy}{dt} = -y$$
(2.149)

The x variable does not appear in the second equation, so we can it solve for $y(t)$ without knowing $x(t)$. We find

$$y(t) = c_1 e^{-t},$$
(2.150)

where c_1 is an arbitrary constant. Then the first equation is

$$\frac{dx}{dt} = -3x + c_1 e^{-t}.$$
(2.151)

This is a linear first order differential equation; it may be solved by using the method discussed in Section 2.5. We find the solution to be

$$x(t) = c_2 e^{-3t} + \frac{c_1}{2} e^{-t}$$
(2.152)

where c_2 is an arbitrary constant. Let's write our solutions (2.150) and (2.152) as a vector:

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_2 e^{-3t} + \frac{c_1}{2} e^{-t} \\ c_1 e^{-t} \end{bmatrix} = c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-3t} + c_1 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} e^{-t}$$
(2.153)

Because the second equation of this example was *decoupled* from the first, we were able to find the solution by solving two first order differential equations. Unfortunately, this method won't work for the general problem. □

Solving the System. To motivate our method for solving (2.148), we first recall that the solution to the single equation

$$\frac{dx}{dt} = ax$$
(2.154)

is $x(t) = ce^{at}$: a constant multiplied by an exponential. We also note that the solution to the example problem (2.149) that is given by (2.153) is a linear combination of functions of the form $\vec{\mathbf{v}}e^{\lambda t}$, where $\vec{\mathbf{v}}$ is constant vector. This suggests that to solve (2.148), we look for a solution of the form

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{v}}e^{\lambda t}$$
(2.155)

where $\vec{\mathbf{v}}$ is a constant vector, and λ is a number. We note that $\vec{\mathbf{x}}(t) = \vec{\mathbf{0}}$ is a solution to (2.148), which we call the *trivial solution*. We would like to find nontrivial solutions, so we assume $\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$.

By substituting our "guess" into the differential equation, we find

$$\lambda \vec{\mathbf{v}} e^{\lambda t} = A\left(\vec{\mathbf{v}} e^{\lambda t}\right) = A\vec{\mathbf{v}} e^{\lambda t} \tag{2.156}$$

or, after canceling $e^{\lambda t}$ and rearranging,

$$A\vec{\mathbf{v}} = \lambda \vec{\mathbf{v}}. \tag{2.157}$$

This is the *eigenvalue problem* for the matrix A. To solve (2.148), we must find the eigenvalues (λ) and the corresponding eigenvectors ($\vec{\mathbf{v}}$) of A.

Recall from linear algebra that an $n \times n$ matrix has at most n eigenvalues, and always has at least one eigenvalue. Thus for our 2×2 system, we expect to find one or two eigenvalues. A "typical" or "generic" 2×2 matrix will have two eigenvalues, and that is the case we will study. The case where there is just one eigenvalue is also important, but we will not pursue it in this course.

An eigenvalue may be real or complex. We note that if λ is a complex eigenvalue of A, then the corresponding eigenvector $\vec{\mathbf{v}}$ must also be complex. Moreover, by taking the complex conjugate of both sides of (2.157), we find

$$\begin{aligned} (A\vec{\mathbf{v}})^* &= (\lambda \vec{\mathbf{v}})^* \\ A\vec{\mathbf{v}}^* &= \lambda^* \vec{\mathbf{v}}^* \end{aligned} \tag{2.158}$$

(where z^* is the complex conjugate of z). The second equation follows from the first by using the multiplicative property of the conjugate $(wz)^* = w^* z^*$, and by noting that A is a real matrix, so $A^* = A$. The second equation says that λ^* is also an eigenvalue, with a corresponding eigenvector $\vec{\mathbf{v}}^*$. The short summary is, for a real matrix A, *complex eigenvalues always occur in complex conjugate pairs*.

Finally, it is not hard to verify that if $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are solutions to (2.148), then so is $c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t)$ for any constants c_1 and c_2. (This is the *principle of superposition* for linear systems.) Equation (2.148) is a pair of coupled first order equations, so we expect the general solution to have two arbitrary constants. In fact, the general solution must have the form

$$\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{x}}_1(t) + c_2\vec{\mathbf{x}}_2(t). \tag{2.159}$$

We'll find the general solution in two cases: real and distinct eigenvalues, and complex eigenvalues.

Real, distinct eigenvalues. Suppose that A has real eigenvalues λ_1 and λ_2, and $\lambda_1 \neq \lambda_2$. Let $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ be corresponding eigenvectors. The general solution is

$$\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2\vec{\mathbf{v}}_2 e^{\lambda_2 t} \tag{2.160}$$

where c_1 and c_2 are arbitrary constants.

Complex eigenvalues. Because the matrix A is real, we know that complex eigenvalues must occur in complex conjugate pairs. Suppose $\lambda_1 = \mu + i\omega$, with eigenvector $\vec{\mathbf{v}}_1 = \vec{\mathbf{a}} + i\vec{\mathbf{b}}$ (where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are real vectors). If we use the formula for real eigenvalues, we obtain

$$\vec{\mathbf{x}} = c_1\vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2\vec{\mathbf{v}}_1^* e^{\lambda_1^* t}. \tag{2.161}$$

This expression is a solution to the differential equations. However, for arbitrary c_1 and c_2, this expression will generally be complex-valued, and we want a *real-valued* solution. We derive such a solution based on the following observation: *If the matrix A has only real elements, and $\vec{\mathbf{x}}(t)$ is a complex solution to the linear system of differential equations, then the real and imaginary parts of $\vec{\mathbf{x}}(t)$ are also solutions to the differential equation.* (We

take the "imaginary part" to be the coefficient of i, so the imaginary part is also a real-valued solution.) This claim is easily verified: Assume $\vec{\mathbf{x}}(t) = \vec{\mathbf{p}}(t) + i\vec{\mathbf{q}}(t)$ is a solution to $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}}$. Then

$$\frac{d\vec{\mathbf{p}}}{dt} + i\frac{d\vec{\mathbf{q}}}{dt} = A(\vec{\mathbf{p}} + i\vec{\mathbf{q}}) = A\vec{\mathbf{p}} + iA\vec{\mathbf{q}} \tag{2.162}$$

Two complex expressions are equal if and only if their real and imaginary parts are equal, so this equation implies

$$\frac{d\vec{\mathbf{p}}}{dt} = A\vec{\mathbf{p}} \quad \text{and} \quad \frac{d\vec{\mathbf{q}}}{dt} = A\vec{\mathbf{q}}. \tag{2.163}$$

In other words, $\vec{\mathbf{p}}(t)$ and $\vec{\mathbf{q}}(t)$ are solutions to the system of differential equations.

Now consider the complex solution

$$\vec{\mathbf{x}}_1(t) = \vec{\mathbf{v}}_1 e^{\lambda_1 t} = (\vec{\mathbf{a}} + i\vec{\mathbf{b}})e^{(\mu + i\omega)t} \tag{2.164}$$

To separate this into its real and imaginary parts, we use *Euler's formula*:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2.165}$$

Then

$$\begin{aligned}
\vec{\mathbf{x}}_1(t) &= \vec{\mathbf{v}}_1 e^{\lambda_1 t} \\
&= (\vec{\mathbf{a}} + i\vec{\mathbf{b}})e^{(\mu + i\omega)t} \\
&= (\vec{\mathbf{a}} + i\vec{\mathbf{b}})e^{\mu t}e^{i\omega t} \\
&= e^{\mu t}(\vec{\mathbf{a}} + i\vec{\mathbf{b}})(\cos\omega t + i\sin\omega t) \\
&= e^{\mu t}\left(\vec{\mathbf{a}}\cos\omega t - \vec{\mathbf{b}}\sin\omega t + i(\vec{\mathbf{a}}\sin\omega t + \vec{\mathbf{b}}\cos\omega t)\right) \\
&= \left[e^{\mu t}\left(\vec{\mathbf{a}}\cos\omega t - \vec{\mathbf{b}}\sin\omega t\right)\right] + i\left[e^{\mu t}\left(\vec{\mathbf{a}}\sin\omega t + \vec{\mathbf{b}}\cos\omega t\right)\right]
\end{aligned} \tag{2.166}$$

The real part of this solution is

$$\vec{\mathbf{u}}(t) = e^{\mu t}\left(\vec{\mathbf{a}}\cos\omega t - \vec{\mathbf{b}}\sin\omega t\right) \tag{2.167}$$

and the imaginary part (i.e. the coefficient of i) is

$$\vec{\mathbf{w}}(t) = e^{\mu t}\left(\vec{\mathbf{a}}\sin\omega t + \vec{\mathbf{b}}\cos\omega t\right). \tag{2.168}$$

Each of these is real-valued, and by the observation given above, each is a solution to the differential equations. We may then write the general solution as

$$\begin{aligned}
\vec{\mathbf{x}}(t) &= c_1\vec{\mathbf{u}}(t) + c_2\vec{\mathbf{w}}(t) \\
&= c_1 e^{\mu t}\left(\vec{\mathbf{a}}\cos\omega t - \vec{\mathbf{b}}\sin\omega t\right) + c_2 e^{\mu t}\left(\vec{\mathbf{a}}\sin\omega t + \vec{\mathbf{b}}\cos\omega t\right).
\end{aligned} \tag{2.169}$$

Note that we never had to write down the second (complex conjugate) eigenvalue or eigenvector.

Classifying the Equilibrium and Its Stability. We now consider several subcases of the real and complex cases. Our goal is to understand the possible behaviors of solutions in the various cases. An important property of an equilibrium point is its *stability*. The following describes three basic cases.

- If all solutions that start close to an equilibrium converge to the equilibrium asymptotically as $t \to \infty$, we say the equilibrium is *asymptotically stable*.

- If all solutions in sufficiently small neighborhoods of the equilibrium remain close to the equilibrium, we say the equilibrium is *stable*. Note that asymptotic stability implies stability.
- If every neighborhood of the equilibrium contains solutions arbitrarily close to the equilibrium that leave the neighborhood, we say the equilibrium is *unstable*.

To determine how to classify the equilibrium, we consider the real and complex cases separately.

Real, distinct eigenvalues. Let λ_1 and λ_2 be real eigenvalues, $\lambda_1 \neq \lambda_2$, and let $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$ be respective eigenvectors. We have the following cases.

(1) $\boldsymbol{\lambda_1 < \lambda_2 < 0}$.

When both eigenvalues are negative, it is clear from (2.160) that

$$\lim_{t \to \infty} \vec{\mathbf{x}}(t) = \vec{\mathbf{0}}.$$

All trajectories approach $\vec{\mathbf{0}}$ asymptotically.

The equilibrium $\vec{\mathbf{0}}$ is called a *sink*. It is *asymptotically stable*.

(2) $\boldsymbol{0 < \lambda_1 < \lambda_2}$.

In this case, all solutions diverge from the origin. However, for any solution $\vec{\mathbf{x}}(t)$, we have

$$\lim_{t \to -\infty} \vec{\mathbf{x}}(t) = \vec{\mathbf{0}},$$

so all solutions converge to $\vec{\mathbf{0}}$ backwards in time.

The equilibrium $\vec{\mathbf{0}}$ is called a *source*. It is *unstable*.

(3) $\boldsymbol{\lambda_1 < 0 < \lambda_2}$.

Looking at (2.160), we see that if $c_2 = 0$, the solution will converge to $\vec{\mathbf{0}}$ as $t \to \infty$. If $c_2 \neq 0$, then eventually the term $c_2 \vec{\mathbf{v}}_2 e^{\lambda_2 t}$ will dominate, and the solution will grow exponentially. Thus, there are solutions that converge to $\vec{\mathbf{0}}$, but "most" (i.e. those with $c_2 \neq 0$) will not.

The equilibrium $\vec{\mathbf{0}}$ is called a *saddle point*, and it is *unstable*.

(4) **Zero eigenvalue**. It is also possible that one of the eigenvalues is zero. The analysis of this case is left as an exercise.

We can say a little more about cases (1) and (2). Consider the sink, where $\lambda_1 < \lambda_2 < 0$. We can write the general solution as

$$\vec{\mathbf{x}}(t) = e^{\lambda_2 t} \left[c_1 \vec{\mathbf{v}}_1 e^{(\lambda_1 - \lambda_2)t} + c_2 \vec{\mathbf{v}}_2 \right]. \tag{2.170}$$

Suppose $c_2 \neq 0$. Since $\lambda_1 - \lambda_2 < 0$, the first term in the square brackets goes to zero as t increases, and for large positive t we have

$$\vec{\mathbf{x}}(t) \approx c_2 \vec{\mathbf{v}}_2 e^{\lambda_2 t}. \tag{2.171}$$

This says that as the solution converges to zero, it does so along the direction of $\vec{\mathbf{v}}_2$. That is, $\vec{\mathbf{x}}(t)$ will approach $\vec{\mathbf{0}}$ along a curve that is tangent to the eigenvector $\vec{\mathbf{v}}_2$.

In the case of a source, where $0 < \lambda_1 < \lambda_2$, we can write the general solution as

$$\vec{\mathbf{x}}(t) = e^{\lambda_1 t} \left[c_1 \vec{\mathbf{v}}_1 + c_2 \vec{\mathbf{v}}_2 e^{(\lambda_2 - \lambda_1)t} \right] \tag{2.172}$$

Suppose $c_1 \neq 0$. For large *negative t*, the second term in the square brackets goes to zero while the first is constant, so we have

$$\vec{\mathbf{x}}(t) \approx c_1 \vec{\mathbf{v}}_1 e^{\lambda_1 t}. \tag{2.173}$$

This says that as $t \to -\infty$, the solution converges to $\vec{0}$ along the direction of $\vec{v}_1$. That is, $\vec{x}(t)$ will approach $\vec{0}$ (as $t \to -\infty$) along a curve that is tangent to the eigenvector $\vec{v}_1$.

We can summarize these observations by saying that in the case of a source or sink, *curved trajectories in the phase plane approach $\vec{0}$ along curves that are tangent to the eigenvector of the eigenvalue closest to zero.* By stating it this way, we don't have to worry about which eigenvalue we called λ_1 or λ_2. This observation will be very useful when we sketch phase portraits.

Complex eigenvalues. We assume one eigenvalue is $\lambda_1 = \mu + i\omega$, and a corresponding eigenvector is $\vec{v}_1 = \vec{a} + i\vec{b}$. We can rewrite the general solution (2.169) as

$$\vec{x}(t) = e^{\mu t} \left[c_1 \left(\vec{a} \cos \omega t - \vec{b} \sin \omega t \right) + c_2 \left(\vec{a} \sin \omega t + \vec{b} \cos \omega t \right) \right] \tag{2.174}$$

Note that the only t dependence in the expression in the square brackets is through $\sin \omega t$ and $\cos \omega t$. Thus the expression in the square brackets is periodic, with period $T = 2\pi/\omega$. All solutions therefore have an oscillatory behavior, but whether the oscillation grows or decays depends on the sign of μ. We have the following subcases.

$\boldsymbol{\mu < 0}$ In this case, the amplitude of the oscillation decays exponentially. The trajectories spiral into the origin as t increases, and $\lim_{t \to \infty} \vec{x}(t) = \vec{0}$.

   The equilibrium $\vec{0}$ is called a *spiral sink.* It is *asymptotically stable.*

$\boldsymbol{\mu = 0}$ Since $e^0 = 1$, the solution is periodic. In the phase plane, the trajectories are ellipses.

   The equilibrium $\vec{0}$ is called a *center*, and it is *stable.*

$\boldsymbol{\mu > 0}$ In this case, the amplitude of the oscillation grows exponentially. The trajectories spiral out of the origin as t increases. Considering backwards time, we have $\lim_{t \to -\infty} \vec{x}(t) = \vec{0}$.

   The equilibrium $\vec{0}$ is called a *spiral source.* It is *unstable.*

■ EXAMPLE 2.12.2   We consider the linear system

$$\frac{dx}{dt} = x + by$$
$$\frac{dy}{dt} = -x - 2y \tag{2.175}$$

In this example, the linear system depends on a parameter b. We will determine how the classification and stability of the equilibrium $\vec{0}$ depends on b.

The coefficient matrix is

$$A = \begin{bmatrix} 1 & b \\ -1 & -2 \end{bmatrix} \tag{2.176}$$

The eigenvalues of this matrix are

$$\lambda_1 = \frac{-1 + \sqrt{9 - 4b}}{2}, \quad \lambda_2 = \frac{-1 - \sqrt{9 - 4b}}{2}. \tag{2.177}$$

We first note that these eigenvalues are complex if

$$9 - 4b < 0 \implies b > \frac{9}{4} \tag{2.178}$$

In this case, we have $\mu = -1/2$, so the equilibrium is a *spiral sink.*

Range of b	Eigenvalues	Classification	Stability
$b < 2$	Real: $\lambda_1 > 0$, $\lambda_2 < 0$	saddle	unstable
$2 < b < 9/4$	Real: $\lambda_1 < 0$, $\lambda_2 < 0$	sink	asymptotically stable
$b > 9/4$	Complex: $\mu < 0$	spiral sink	asymptotically stable

TABLE 5. Results for Example 2.12.

Next we note that $\lambda_2 < 0$ for any $b < 9/4$. The classification of $\vec{\mathbf{0}}$ depends on the sign of λ_1. We'll find where $\lambda_1 = 0$:

$$\lambda_1 = 0 \implies \frac{-1 + \sqrt{9 - 4b}}{2} = 0 \implies b = 2. \tag{2.179}$$

We have $\lambda_1 < 0$ if $2 < b < 9/4$, and $\lambda_1 > 0$ if $b < 2$. Table 5 summarizes the results. $\quad\square$

Sketching Phase Portraits for Planar Linear Systems. The *phase portrait* is obtained by plotting several representative solutions in the (x, y) plane. Generally, enough solutions are plotted to illustrate the behavior of all the possible solutions to the system.

We summarize the procedure for sketching the phase portraits for the linear system

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \text{where} \quad \vec{\mathbf{x}} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Procedure

(1) Find the eigenvalues of the matrix, and classify the equilibrium as a saddle, sink, source, spiral source, spiral sink, or center. (There are a few other special cases that we did not cover.)

(2) If the eigenvalues are real, find the associated eigenvectors, and draw the straight-line solutions.

(3) Sketch the *nullclines*, and indicate the directions of the vector field on the null-clines.

The *x nullcline* is the line where $\frac{dx}{dt} = 0$, so $ax + by = 0$. Along this line, the vector field is vertical, so trajectories must have a vertical tangent when they cross this line.

The *y nullcline* is the line where $\frac{dy}{dt} = 0$, so $cx + dy = 0$. Along this line, the vector field is horizontal, so trajectories must have a horizontal tangent when they cross this line.

(4) Indicate the direction of the vector field on the x and y axes. Note that at the points $(1, 0)$ and $(0, 1)$, the vector field is given by the first and second columns of A, respectively.

(5) If the eigenvalues are real, distinct, and of the same sign, all trajectories except the straight-line solutions will approach the origin *tangent to the eigenvector of the eigenvalue closest to zero*.

(6) Use the above information to sketch several representative trajectories.

The following examples demonstrate this procedure. (In the examples, the general solution will be found in step (2), even though this isn't strictly necessary if you only want to sketch the phase portrait.)

■ EXAMPLE 2.12.3

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}}, \quad \text{where} \quad A = \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & -1 \end{bmatrix}$$

or equivalently,

$$\frac{dx}{dt} = \frac{1}{2}x - y$$

$$\frac{dy}{dt} = x - y$$

(1) The characteristic polynomial is $\lambda^2 + \frac{1}{2}\lambda + \frac{1}{2}$, and the eigenvalues are $\lambda = -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$. The eigenvalues are complex, and the real part is negative, so the origin is a *spiral sink*.

(2) In the complex case, we don't need the eigenvectors to sketch the phase portrait, but we do need an eigenvector $\vec{\mathbf{v}}_1$ associated with $\lambda_1 = \mu + i\omega$ to find the general solution. In this case, $\lambda_1 = -\frac{1}{4} + i\frac{\sqrt{7}}{4}$, so $\mu = -\frac{1}{4}$ and $\omega = \frac{\sqrt{7}}{4}$, and

$$A - \lambda_1 I = \begin{bmatrix} \frac{1}{2} - \left(-\frac{1}{4} + i\frac{\sqrt{7}}{4}\right) & -1 \\ 1 & -1 - \left(-\frac{1}{4} + i\frac{\sqrt{7}}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{3}{4} - i\frac{\sqrt{7}}{4} & -1 \\ 1 & -\frac{3}{4} - i\frac{\sqrt{7}}{4} \end{bmatrix} \quad (2.180)$$

From the first row of $A - \lambda_1 I$ we see that we can choose

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ \frac{3}{4} - i\frac{\sqrt{7}}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} + i\begin{bmatrix} 0 \\ -\frac{\sqrt{7}}{4} \end{bmatrix} \quad (2.181)$$

So

$$\vec{\mathbf{a}} = \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{b}} = \begin{bmatrix} 0 \\ -\frac{\sqrt{7}}{4} \end{bmatrix} \quad (2.182)$$

The general solution is then

$$\vec{\mathbf{x}}(t) = c_1 e^{-(1/4)t} \left\{ \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \cos\left(\sqrt{7}t/4\right) - \begin{bmatrix} 0 \\ -\frac{\sqrt{7}}{4} \end{bmatrix} \sin\left(\sqrt{7}t/4\right) \right\}$$

$$+ c_2 e^{-(1/4)t} \left\{ \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} \sin\left(\sqrt{7}t/4\right) + \begin{bmatrix} 0 \\ -\frac{\sqrt{7}}{4} \end{bmatrix} \cos\left(\sqrt{7}t/4\right) \right\} \quad (2.183)$$

(3) The x nullcline is

$$\frac{1}{2}x - y = 0, \quad \text{or} \quad y = \frac{1}{2}x.$$

At the point $(1, 1/2)$, we have $\frac{dy}{dt} = 1/2 > 0$, so the vectors on the x nullcline in the first quadrant point in the positive y direction. (Since this problem is linear, they must point in the opposite direction in the opposite quadrant.)

The y nullcline is

$$x - y = 0, \quad \text{or} \quad y = x.$$

At the point $(1, 1)$, we have $\frac{dx}{dt} = -1/2$, so the vectors on the y nullcline in the first quadrant point in the negative x direction.

(4) The vector field at $(1, 0)$ is $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$, and at $(0, 1)$ it is $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Vectors in these directions are drawn on the positive x and y axes, respectively (and vector in the opposite directions are drawn on the negative x and y axes).

(5) (The eigenvalues are complex, so this guideline is not applicable.)

(6) Here is the phase portrait. The dashed lines are the nullclines. We know the
equilibrium is a spiral sink, so the trajectories will spiral into the origin. We use
the vector field on the nullclines and the axes to make a "pretty good" sketch of
two of the spiral trajectories.

□
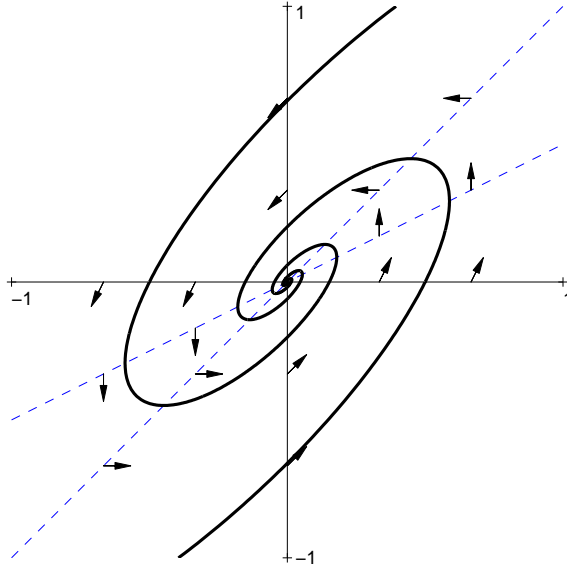
■ EXAMPLE 2.12.4   We consider the linear system of differential equations with

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -3 \end{bmatrix}$$

(1) The characteristic polynomial is $\lambda^2 - \lambda - 6$, and the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. We have real eigenvalues with opposite signs, so $\vec{0}$ is a *saddle point*.

(2) The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. These will correspond to straight-line solutions with slope 3 and $1/2$, respectively, in the phase plane.

The general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t}. \tag{2.184}$$

(3) The x nullcline is $4x - 2y = 0$, or $y = 2x$, and the y nullcline is $3x - 3y = 0$, or $y = x$. These are shown as dashed lines in the phase plane.

(4) The vector field at $(1,0)$ is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and the vector field at $(0,1)$ is $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$. Vectors in the same direction as these are shown on the positive x and y axes, respectively. (They are in the opposite directions on the negative axes.

(5) (This guideline is not applicable, since the eigenvalues have opposite signs.)

(6) Here is the phase portrait:

□
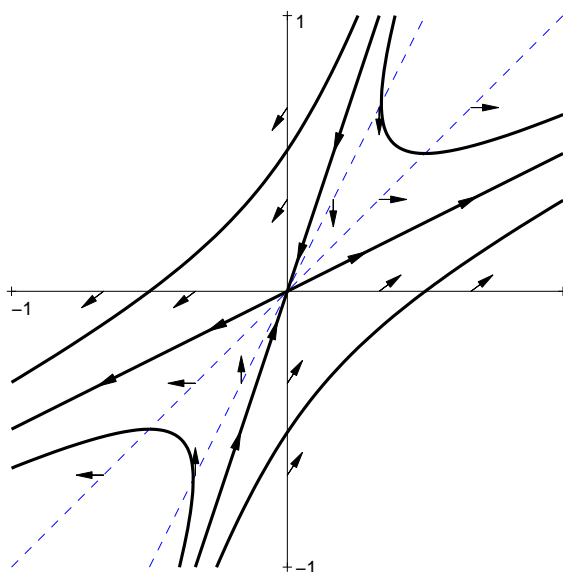
■ EXAMPLE 2.12.5   We'll look at one more example, this time with

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

(1) The characteristic polynomial is $\lambda^2 - 7\lambda + 10$, and the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$. We have positive real eigenvalues, so $\vec{0}$ is a *source*.

(2) The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. These will correspond to straight-line solutions with slope -1 and $1/2$, respectively, in the phase plane.

The general solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}. \tag{2.185}$$

(3) The x nullcline is $4x + 2y = 0$, or $y = -2x$, and the y nullcline is $x + 3y = 0$, or $y = -x/3$. These are the dashed lines in the phase plane.

(4) The vector field at $(1,0)$ is $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and the vector field at $(0,1)$ is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Vectors in the same direction as these are shown on the positive x and y axes, respectively.

(5) We have two distinct eigenvalues with the same sign, so we know that the equilibrium is a source, and all trajectories (except the straight-line solution associated with the bigger eigenvalue) will come out of the origin tangent to the eigenvector associated with the eigenvalue closest to zero. In this case, the trajectories will be tangent to the direction of $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(6) Here is the phase portrait:

□
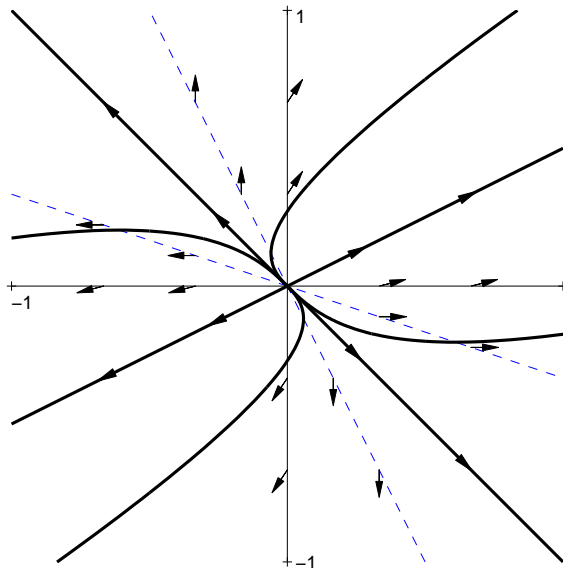
Exercises

2.12.1. Follow the procedure outlined in this section (and demonstrated in the examples) to sketch the phase portraits for the linear systems with the following coefficient matrices.

(a) $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$
(b) $A = \begin{bmatrix} -2 & 3 \\ -2 & -4 \end{bmatrix}$
(c) $A = \begin{bmatrix} -4 & -1/2 \\ -1 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 3 & 1 \\ -1 & -3 \end{bmatrix}$
(e) $A = \begin{bmatrix} 0 & 6 \\ 3/2 & 0 \end{bmatrix}$
(f) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1/2 \end{bmatrix}$

2.12.2. Suppose A has one zero eigenvalue and one nonzero eigenvalue.

(a) What is the general solution?
(b) Describe the set of equilibrium solutions.
(c) Describe the phase portrait.

2.12.3. Each of the following matrices has one zero eigenvalue. Find the general solution of the corresponding linear system of differential equations, and sketch the phase portrait.

(a) $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$
(b) $A = \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$
(c) $A = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$