Math 312 Lecture Notes
Modeling

Warren Weckesser
Department of Mathematics
Colgate University

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Classifying Mathematical Models – An Example

We consider the following scenario. During a storm, a large tree with several mice is blown into the ocean. The storm carries the tree many miles until it washes ashore on an island that, until now, has had no mice. This island has many seed-bearing plants that mice love, and a nice climate, so the mice have a good chance to survive and prosper.

How will the population of mice on this island change over time?

For simplicity, we will consider just the population of female mice. We will also assume that a new generation is produced each year. We begin by making the following assumptions

1. In each generation, each female produces three offspring (along with some number of males).
2. The offspring can reproduce after one year.
3. Mice live forever.

Clearly these are not realistic assumptions. We will accept them for now, in order to develop a simple model. Later we will look at some more realistic variations.

To start, we will work with *discrete time*. Let $p(n)$ be the population at the end of the nth year, where n is an integer. With this notation, $p(0)$ is the initial population. Let's suppose that $p(0) = 1$; that is, the initial population contained just one female mouse.

At the end of the first year, this mouse has produced three female offspring, so $p(1) = 4$. At the end of the second year, each of the four mice has produced three more offspring, so $p(2) = 16$. In general, we have

$$p(n+1) = 4p(n) \tag{1}$$

Equation (1) is the rule that describes how the population changes over time. Such a rule involving discrete time is sometimes called a *difference equation*. This terminology is a bit clearer if we rewrite the equation as

$$p(n+1) - p(n) = 3p(n) \tag{2}$$

This gives the rule for computing the difference between successive generations.

We can easily verify that (1) has the solution

$$p(n) = p(0)4^n \tag{3}$$

This is a solution in the sense that the population at time n is given directly as a function of n and the initial population.

If we plot this solution, we will see a "stair-step" plot, with the size of the steps getting larger as n increases. If all the mice produce their offspring at exactly the same time, then this stair-step shape is reasonable. But we don't really expect that to be the case. Presumably mice will be born throughout the year, and we expect the actual graph of the population to have many smaller steps. In fact, when the population is large (and if we blur our vision a bit), we might expect the graph to look like a smooth curve. Let $p(t)$ be the population at time t, where now t is a real number. What mathematical rule is obeyed by $p(t)$?

If we still believe our assumptions, we still expect that in one year, the population increases four-fold. That is, we still have

$$p(t+1) = 4p(t) \tag{4}$$

What is the corresponding rule for increments of time less than one year? That is, what can we say about $p(t+\frac{1}{2})$, $p(t+\frac{1}{3})$, or in general, $p(t+h)$? I claim that the correct rule is

$$p(t+h) = 4^h p(t) \tag{5}$$

If $h = 0$, we obtain $p(t) = p(t)$, as we should, and if $h = 1$, we obtain (4). If $h = 1/k$, where k is an integer, we have

$$\begin{aligned}
p(t+1) &= p(t + \frac{k-1}{k} + \frac{1}{k}) \\
&= 4^{1/k} p(t + \frac{k-1}{k}) \\
&= 4^{2/k} p(t + \frac{k-2}{k}) \\
&\vdots \\
&= 4^{\frac{k-1}{k}} p(t + \frac{1}{k}) \\
&= 4p(t)
\end{aligned} \tag{6}$$

so the repeated application of (5) with $h = 1/k$ also agrees with (4).

We now subtract $p(t)$ from both sides of (5), and divide by h:

$$\frac{p(t+h) - p(t)}{h} = \frac{4^h p(t) - p(t)}{h} = \left(\frac{4^h - 1}{h} \right) p(t). \tag{7}$$

Take the limit $h \to 0$. On the left we obtain $p'(t)$. On the right, we apply L'Hopital's Rule (and recall that $\frac{d}{dx}[a^x] = \ln(a)a^x$) to obtain $\ln(4)p(t)$. Thus we have

$$p'(t) = \ln(4)p(t) \tag{8}$$

This is a *differential equation*. This equation says that the instantaneous rate of change of the population at time t is proportional to the population at time t; the proportionality constant is $\ln(4)$.

Equations (1) and (8) both give rules for determining the population. The first is a *discrete time* model, and the second is a *continuous time* model. This distinction is one of the fundamental categorizations of models.

We now consider a more complicated discrete model, in which we no longer assume that the mice live forever. Suppose the mice only live three years. Moreover, each female mouse produces two female offspring during its second year and its third year. At time n, we need *three* quantities to describe the state of the population. We define

- $p_0(n)$ is the number of new female offspring in year n;
- $p_1(n)$ is the number of one-year-old females in year n; and
- $p_2(n)$ is the number of two-year-old females in year n.

Then we have

$$\begin{aligned} p_0(n+1) &= 2p_1(n) + 2p_2(n) \\ p_1(n+1) &= p_0(n) \\ p_2(n+1) &= p_1(n) \end{aligned} \tag{9}$$

Suppose that in the inital population, $p_0(0) = 0$, $p_1(0) = 2$ and $p_2(0) = 0$. Let's compute the population for a few generations:

n	$p_0(n)$	$p_1(n)$	$p_2(n)$	Total
0	0	2	0	2
1	4	0	2	6
2	4	4	0	8
3	8	4	4	16
4	16	8	4	28

We appear to have a growing population, but unlike the simpler model, a formula for the solution is not obvious. (We will see how to solve a problem like this later in the course.)

A key observation to make about the model is that the "state" of the population is three dimensional. In order to write down the rules that determine how the population changes, we needed to keep track of three quantities. We can put these in a vector:

$$\vec{\mathbf{x}}(n) = \begin{bmatrix} p_0(n) \\ p_1(n) \\ p_2(n) \end{bmatrix} \tag{10}$$

Then the rules given in (9) can be written more concisely as

$$\vec{\mathbf{x}}(n+1) = \vec{\mathbf{f}}(\vec{\mathbf{x}}(n)) \tag{11}$$

where $\vec{\mathbf{f}}$ is the vector-valued function (or *map*, or *mapping*) given by the right side of (9).

Equation (11) is a general form for multi-dimensional discrete time models. We will often call such an equation an *iterated map*. (Just like the one-dimensional case, these are also often called *difference equations*.)

We will also study multi-dimensional *systems of differential equations*:

$$\vec{\mathbf{x}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{x}}(t)) \tag{12}$$

where

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_m(t) \end{bmatrix} \quad \text{and} \quad \vec{\mathbf{x}}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_m'(t) \end{bmatrix} \tag{13}$$

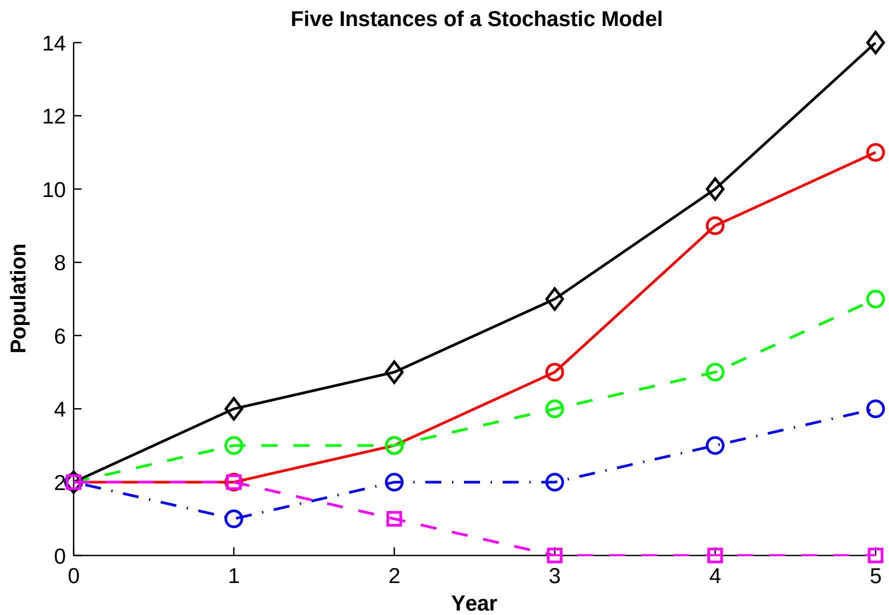

Figure 1: Five instances of the population in a stochastic model.

Deterministic vs. Stochastic Models. We have one more important categorization to discuss. Both (11) and (12) are *deterministic*. That is, for a given starting state (e.g. $\vec{\mathbf{x}}(0)$), the fate of the population is determined; the equations produce only one possible solution. There are no random events incorporated in the model.

Models that explicitly include random events are called *stochastic*. For example, suppose we model a population of mice with the rule that in each year, there is a one in ten chance that a mouse will die, and a one in two chance that the mouse will produce one offspring. (This means there is a four in ten chance that the mouse will not die and will not produce an offspring.) In such a model, if the initial population is 1, then in the next year the population could be 0, 1, or 2. The following year it could be 0, 1, 2, 3 or 4. Figure 1 shows five instances of the population for this model. In all five cases, the initial population is 2.

In such models, the question that we ask is not "What is the population in year n?" Rather, we usually ask "What is the probability distribution of the population in year n?" If we have the probability distribution at year n, we can then answer questions such as "What is the probability that the population is zero?" or "What is the probability that the population is greater than 500 in year 10?"

Figure 2 shows a numerically computed distribution of the population after five years. This was computed by running 50000 simulations, and adding up the number of times each possible final population occurred. For example, in the figure we see that approximately 3.6% of the time, the population is zero after five years.

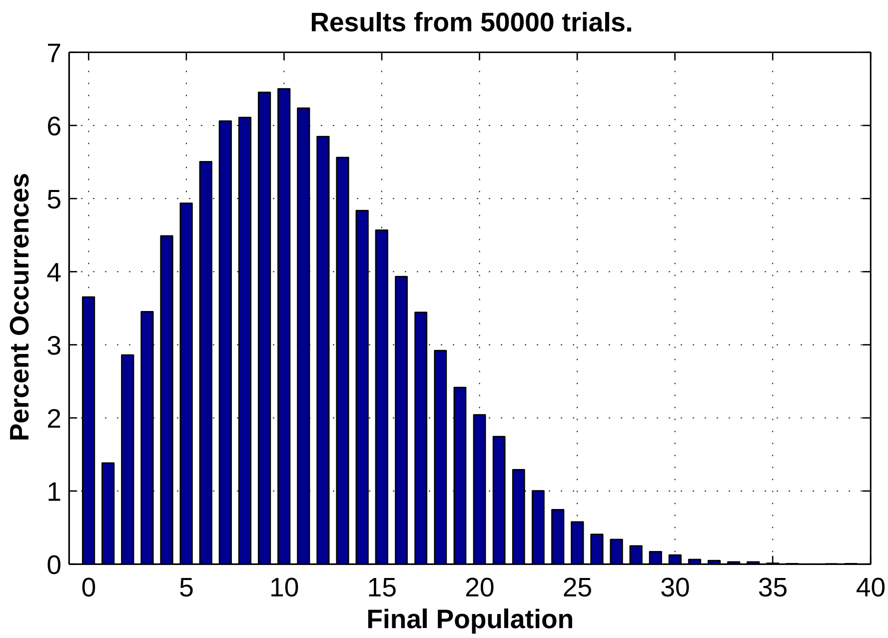

Figure 2: Numerically computed population probability distribution for the population after five years. This was computed by tallying the results of 50000 individual simulations.