4.1. Solow's Economic Growth Model

(Draft version[1]*.)*

We consider a model from macroeconomics. Let K be the capital,[2] L the labor, and Q the production output of an economy. We are interested in a *dynamic* problem, so $K(t)$, $L(t)$ and $Q(t)$ are all functions of time, but we will suppress the t argument. In elementary economics, one learns that a common assumption is that Q can be expressed as function of K and L:

$$Q = f(K, L). \tag{4.1}$$

We assume that f has, using economics terminology, *constant returns to scale*. Mathematically, this means that multiplying K and L by the same amount results in Q being multiplied by the same amount. That is, for any constant b,

$$f(bK, bL) = bf(K, L). \tag{4.2}$$

For example, the Cobb-Douglas function $f(K, L) = K^{1/3}L^{2/3}$ satisfies this assumption.

We make two more assumptions. We assume that a constant proportion of Q is invested in capital. This means that the *rate of change* of K is proportional to Q:

$$\frac{dK}{dt} = sQ, \tag{4.3}$$

where $s > 0$ is the proportionality constant. We also assume that the labor force is growing according to the equation

$$\frac{dL}{dt} = \lambda L, \tag{4.4}$$

where $\lambda > 0$ is the per capita growth rate. This is a first order equation for L which we can solve to find $L = L_0 e^{\lambda t}$.

If possible, we would like to combine (4.1), (4.3), and (4.4) into a single equation that we may easily analyze. A natural first attempt is to substitute (4.1) into (4.3) to obtain

$$\frac{dK}{dt} = sf(K, L) \tag{4.5}$$

Since $L(t)$ is a known function, the only unknown function is $K(t)$. Thus this is a first order differential equation for $K(t)$. It is, however, nonautonomous. $L(t) = L_0 e^{\lambda t}$, so the right side depends on t explicitly. We could still try to analyze this equation, but it would be nice if we could find an *autonomous* first order differential equation. It turns out we can derive an autonomous equation for the *ratio* $\frac{K}{L}$ instead of K.

First, because f has constant returns to scale, we may write

$$f(K, L) = f\left(L\frac{K}{L}, L\right) = Lf\left(\frac{K}{L}, 1\right). \tag{4.6}$$

Then, after dividing by L, (4.5) becomes

$$\frac{1}{L}\frac{dK}{dt} = sf\left(\frac{K}{L}, 1\right) \tag{4.7}$$

---

[2]*Capital* includes things that are owned to be used in production, such as buildings and manufacturing equipment.

Next, we consider the derivative of $\frac{K}{L}$ given by the quotient rule, and we use (4.4):

$$\frac{d}{dt}\left(\frac{K}{L}\right) = \frac{1}{L}\frac{dK}{dt} - \frac{K}{L^2}\frac{dL}{dt} = \frac{1}{L}\frac{dK}{dt} - \lambda\frac{K}{L}. \tag{4.8}$$

If we subtract $\lambda\frac{K}{L}$ from both sides of (4.7), the left side becomes $\frac{d}{dt}\left(\frac{K}{L}\right)$, so we obtain

$$\frac{d}{dt}\left(\frac{K}{L}\right) = sf\left(\frac{K}{L}, 1\right) - \lambda\frac{K}{L} \tag{4.9}$$

We now have an equation in which the unknown function is $\frac{K}{L}$. Let us define

$$k = \frac{K}{L} \tag{4.10}$$

and

$$g(k) = f(k, 1). \tag{4.11}$$

Then (4.9) becomes

$$\frac{dk}{dt} = sg(k) - \lambda k \tag{4.12}$$

This is the *Solow Growth Model* [**8**] which models the growth of the ratio of capital to labor under the assumptions given earlier.

Summary

Assumptions

(1) $Q = f(K, L)$ where $f(K, L)$ is a function with constant returns to scale.

(2) $\frac{dK}{dt} = sQ$; a fraction of the production output is invested in capital.

(3) $\frac{dL}{dt} = \lambda L$; labor grows according to this equation.

Definitions

- $k = \frac{K}{L}$; we analyze the *ratio* of capital to labor.
- $g(k) = f(k, 1)$.

Result

$$\frac{dk}{dt} = sg(k) - \lambda k$$

Example 4.1.1. As an example, let's take the production function to be

$$f(K, L) = K^{1/3}L^{2/3}. \tag{4.13}$$

Then

$$g(k) = f(k, 1) = k^{1/3}, \tag{4.14}$$

and the differential equation for k is

$$\frac{dk}{dt} = sk^{1/3} - \lambda k. \tag{4.15}$$

Figure 4.1 shows the graph of $\frac{dk}{dt}$ versus k.

By solving

$$sk^{1/3} - \lambda k = 0,$$

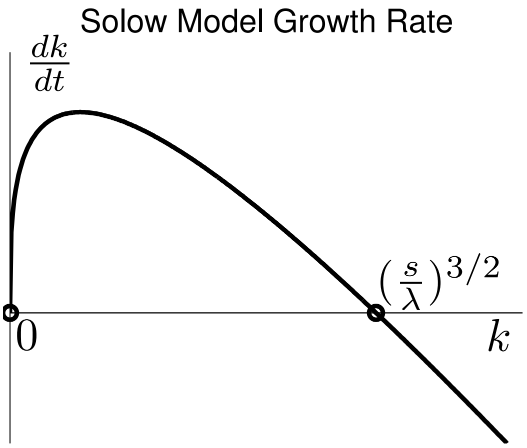

FIGURE 4.1. The graph of the right side of equation (4.15).

we find the equilibrium solutions to be $k = 0$ or $k = (s/\lambda)^{3/2}$.

Changing λ or s will change the scale (and the numerical value of the non-zero equilibrium), but the graph of dk/dt versus k will always have the same qualitative shape as the graph shown above.

We see that if $k > 0$ is small, $\frac{dk}{dt} > 0$, so k will increase; the equilibrium $k = 0$ is *unstable*. The graph of $k(t)$ will have an inflection point when k reaches $\left(\frac{s}{3\lambda}\right)^{3/2}$ (where right side of (4.15) has its maximum). k will then converge asymptotically to the non-zero equilibrium.

The equilibrium $k = (s/\lambda)^{3/2}$ is *asymptotically stable*: any solution that starts near the equilibrium will converge to the equilibrium as $t \to \infty$. In fact, *all* solutions with $k(0) > 0$ will converge asymptotically to this equilibrium.

What does this mean in terms of the capital K and the labor L? Since $k(t) = K(t)/L(t)$, and $L(t) = L_0 e^{\lambda t}$, if $k(t)$ converges to an asymptotically stable equilibrium k_1, then $K(t)$ must behave asymptotically like $k_1 L(t)$. This means that, in the long term, $K(t)$ must grow exponentially, with the same exponent as $L(t)$. This model predicts that in the long term, capital will grow exponentially along with the labor. If, for example, the capital is too low, it will rapidly increase until it becomes approximately proportional to the labor, and then it will settle into a long term behavior in which capital remains proportional to the labor.

Exercises

4.1.1. Verify that the maximum value of the right side of (4.15) occurs at $k = \left(\frac{s}{3\lambda}\right)^{3/2}$.

4.1.2. Find an explicit solution for (4.15), assuming that the initial condition is $k(0) = k_0 > 0$. Use your solution to verify analytically that $\lim_{t\to\infty} k(t) = (s/\lambda)^{3/2}$.

4.1.3. Suppose that labor grows according to a logistic equation

$$\frac{dL}{dt} = \lambda L \left(1 - \frac{L}{M}\right) \tag{4.16}$$

where M is the carrying capacity for the labor population. Derive a new differential equation for $k(t)$. Is your new equation autonomous? If not, can you still determine the asymptotic behavior of the solutions as $t \to \infty$?

4.1.4. Suppose we include the fact that capital deteriorates over time. We replace the assumption given in equation (4.3) with

$$\frac{dK}{dt} = -rK + sQ \tag{4.17}$$

where $r > 0$. (This says that, if $Q = 0$, then K will decay and approach zero asymptotically.) Show that this leads to the differential equation

$$\frac{dk}{dt} = sg(k) - (r + \lambda)k, \tag{4.18}$$

where k and $g(k)$ are defined in (4.10) and (4.11), respectively. Find the equilibrium solutions of this model, and describe the behavior of all possible solutions (assumging $k(0) > 0$). Compare to the Solow model.
