Math 312 **Spring 2006**

Homework Assignment 3

Due *Friday, February 10.*

1. We consider a modification of the SI model. Suppose that infectives eventually recover from the disease, and once they have recovered, they are immune. The recovered individuals no longer spread the disease, and they can not catch it again. We now have three classes of individuals: susceptible (S), infective (I), and recovered (R).

   To make a set of differential equations that models this process, we need to know something about the recovery process. We assume that in a short period of time, a certain fraction of the infectives will recover. The infinitesimal version of this assumption is that *the recovery rate of the infectives is proportional to the number of infectives.*

   The recovery process converts infective individuals into recovered individuals, so this assumption also gives us the differential equation for R.

   (a) Write down a system of three differential equations that models this process. Introduce proportionality constants as needed. Then, compare your answer to the equations given at

   `http://math.colgate.edu/~wweckesser/solver/DiseaseSIR.shtml`

   (The constant γ on that web page is not the same γ mentioned in the lecture notes about the SI model. In the equations on the web page, γ is the "per capita recovery rate of the infectives." Also, on my web browser, the symbol γ looks more like a "Y"; not all web browsers do a good job rendering Greek letters.)

   (b) Let N be the total population (so $N = S + I + R$). Use the differential equations to verify that N remains constant.

   (c) The web page mentioned above allows you to solve the system of differential equations. (Actually, the "solution" is an approximation generated by a computer program, but the approximation is very accurate in most cases.) Leave the default values in the input fields, and click on "Show Solution". Print and hand in a copy of the result.

   You will notice two important differences between this model and the SI model:

   i. $I(t)$ shows a "bump", but then it appears that $I(t)$ approaches zero after the bump.

   ii. $S(t)$ decreases monotonically, but $S(t)$ levels off at a non-zero value. In this model, it appears that not everyone becomes infected.

   (d) The previous part suggests that $I(t)$ will approach zero asymptotically. Use the differential equations to give a convincing argument that this is, in fact, the case: for any positive initial conditions (and assuming that all the proportionality constants are positive), $\lim_{t\to\infty} I(t) = 0$.

   (Be careful that you don't make the following common mistake. Suppose there is a function $x(t)$, and you know that $\frac{dx}{dt} < 0$ whenever $x > 0$. The common mistake is to conclude that $x(t)$ must decrease to *zero* asymptotically. But in fact, $x(t)$ might approach a nonzero constant asymptotically. For example, look again at the plot you made in part (c). Note that $S(t)$ is always decreasing, but $S(t)$ does not approach zero.)

   (e) Use the web page again, restore the default values (in particular, $r = 2$), increase the duration of the solution to 50, and then (by "trial and error") find a value of γ for which half the population never becomes infected. Find the value of γ accurate to three significant digits. Hand in a plot of the solution.

   (f) Repeat part (e) two more times, but use $r = 1.5$ and then $r = 1.0$ instead of the default value of r. (Yes, this is tedious, but it is not difficult, and there is a point to the exercise that will be made in the next homework.) Make a plot of γ vs. r, and conjecture what the actual relationship might be.