**Math 312** **Spring 2006**

## Homework Assignment 4

Due *Friday, February 24.*

1. Consider the first order differential equation

$$\frac{dp}{dt} = r\left(1 - \frac{p}{K}\right)p - ap,$$

where $a \geq 0$ is a constant. This is a modification of the logistic equation.

   (a) If $p$ represents the population of some animal, give a possible interpretation of the $-ap$ term. (Hint: Review Problem 5 of Homework 1. That was a problem in which time was discrete, but the modeling questions are related.)

   (b) Note that the dimensions of both $r$ and $a$ are $\mathcal{T}^{-1}$ (i.e. *(time)*$^{-1}$), so either of these parameters could be used to create a nondimensional time variable $\tau$. Find the corresponding nondimensional differential equations that results for each choice. Be sure to express each equation using only a nondimensional parameter (or parameters) that you define in terms of the given dimensional parameters. How are the two nondimensional differential equations related?

   (c) Assume $p(0) = p_0 > 0$. How does the long term behavior of $p(t)$ depend on the parameters $a$ and $p_0$? Are there positive values of $a$ for which the population can become extinct?

2. The SIR model of the previous homework is

$$\frac{dS}{dt} = -rSI$$
$$\frac{dI}{dt} = rSI - \gamma I$$
$$\frac{dR}{dt} = \gamma I$$

with initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0$$

Counting the initial conditions, this problem has five parameters.

   (a) Use the procedure described in class to nondimensionalize these equations. Since $\gamma$ has dimension *(time)*$^{-1}$, this parameter provides a natural choice for the scaling of the nondimensional time variable $\tau$. For $S$, $I$, and $R$, there are several choices for the rescaling. For this problem, use $S_0 + I_0 + R_0$ as the combination of parameters with which you define the nondimensional versions of $S$, $I$, and $R$. (That is, define, say, $x = S/(R_0 + S_0 + I_0)$, $y = I/(R_0 + S_0 + I_0)$, etc.) Your final answer should be a system of differential equations and initial conditions that contain only nondimensional variables and parameters. How many nondimensional parameters are there in your answer?

   (You might find it convenient to define $N = S_0 + I_0 + R_0$. If you use $x$, $y$ and $z$ for the nondimensional versions of $S$, $I$ and $R$, respectively, you'll then end up with the initial conditions $x_0$, $y_0$ and $z_0$, where $x_0 = S_0/N$, etc. With these definitions, $x_0 + y_0 + z_0 = 1$, so $x_0$, $y_0$ and $z_0$ are not independent parameters. Once $x_0$ and $y_0$ are known, we must have $z_0 = 1 - x_0 - y_0$, so only $x_0$ and $y_0$ are true parameters that can be independently controlled. Take this into account when you count the number of parameters in your answer.)

(b) In Problem 1(f) of Homework 3, conditions under which half the population became infected (for the given initial conditions) appeared to be a linear relation $\gamma = kr$ where $k$ is a constant. (In fact, $k \approx 0.725$.) Use the result of 2(a) above to prove this result.

(c) Repeat (a), but nondimensionalize $S$, $I$ and $R$ using a combination of only $r$ and $\gamma$.

3. A variation of the $SIR$ model is the $SIQR$ model, in which some of the infectives are quarantined, so they no longer react with the susceptibles. More precisely, we assume that *the rate at which infected individuals are quarantined is proportional to the size of the infected population.*

   To create a new set of differential equations, we introduce a new variable, $Q(t)$, to represent the size of the quarantined population. Since the quarantined population are infected, they recover by the same mechanism as the infected population.

   (a) Write down a system of differential equations for the SIQR model. Define any parameters that you use.

   (b) Verify that $N = S + I + Q + R$ is constant.

   (c) Find the *nondimensional* SIQR equations. Define the nondimensional parameters that you use in the equations.