Math 312 - Applied Math: Social Sciences **Spring 2006**

Homework Assignment 5

Due *Friday, March 3*

1. Do Exercise 3.4.2 in the Lanchester Model lecture notes. (The notes are available on the Math 312 web page.)

2. We consider a naive model of a relationship between "Romeo" and "Juliet". We assume that Juliet's attraction to Romeo increases when Romeo is in love with Juliet. Romeo, on the other hand, becomes most interested in Juliet when she is not interested in him.

   To create a mathematical model, we assume that the degree of affection of Juliet for Romeo can be represented as a number $J(t)$. Positive values of J correspond to affection, while negative values correspond to antipathy. Similarly, we use $R(t)$ to describe the degree of affection of Romeo for Juliet.

   We translate our initial assumptions about Romeo and Juliet as follows:

   - The rate of change of Juliet's affection for Romeo is proportional to Romeo's affection for Juliet. That is,
   $$\frac{dJ}{dt} = k_1 R \quad \text{where } k_1 > 0.$$
   - The rate of change of Romeo's affection for Juliet is proportional to Juliet's affection for Romeo, but with the opposite sign. That is
   $$\frac{dR}{dt} = -k_2 J \quad \text{where } k_2 > 0.$$

   Let's suppose that the constants are $k_1 = 2$ and $k_2 = 1$. We then have the linear system of differential equations

   $$\begin{aligned} \frac{dR}{dt} &= -J \\ \frac{dJ}{dt} &= 2R \end{aligned} \tag{1}$$

   In vector and matrix form, this equation is

   $$\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix} \vec{\mathbf{x}}, \quad \text{where} \quad \vec{\mathbf{x}} = \begin{bmatrix} R \\ J \end{bmatrix} \tag{2}$$

   (a) Find the general solution.

   (b) Solve the initial value problem $\vec{\mathbf{x}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

   (c) Use the PPLANE Java applet to plot a phase portrait for this system. Hand in a printed copy of the plot. (The PPLANE web page is **math.rice.edu/~dfield/dfpp.html**. Click on the PPLANE button. Your web browser must have Java installed.)

(d) Classify the equilibrium $\vec{\mathbf{0}}$ as either a saddle, source, sink, spiral source, spiral sink, or center, and determine the stability of the equilibrium $\vec{\mathbf{0}}$. (That is, state whether it is asymptotically stable, stable or unstable.)

3. Now consider the modification of the Romeo and Juliet model in which the emotional states of Romeo and Juliet diminish with time if they are not interacting with each other. In the model of Problem 1, if Romeo suddenly disappeared (i.e if R remained 0 forever), Juliet's emotional state would be governed by $dJ/dt = 0$, which means she would be stuck in whatever state she happened to be in when Romeo vanished. A more realistic (?) model would have Juliet's love for Romeo diminish if Romeo is not around. This suggests adding the term $-\alpha J$ to the formula for dJ/dt, where α is a positive constant. We'll also assume that, in the absence of Juliet, Romeo's love for Juliet also decays. For simplicity, let's assume that Romeo's love fades twice as fast as Juliet's, so the term we add to dR/dt is $-2\alpha R$. The revised model is

$$\begin{aligned}\frac{dR}{dt} &= -2\alpha R - J \\ \frac{dJ}{dt} &= 2R - \alpha J\end{aligned} \tag{3}$$

(a) Assuming only that $\alpha > 0$, determine how the classification of the equilibrium $(0, 0)$ depends on α, and show that $(0, 0)$ is asymptotically stable for all $\alpha > 0$. What does your result say about the long-term romance between Romeo and Juliet?

(b) Use the PPLANE Java applet to plot a phase portrait when $\alpha = 0.1$. Hand in a copy of the plot.

(c) Find the general solution to this linear system of differential equations if $\alpha = 2$. Also, use the PPLANE Java applet to make and print a phase portrait.

(d) Repeat part (c) with $\alpha = 3$.

4. We consider a further modification of the Romeo and Juliet model.

We modify (3) to include constant terms a_R and a_J that account for the inherent "appeal" of Romeo and Juliet, respectively. Unlike α, these parameters are not necessarily positive.

The new differential equations are

$$\begin{aligned}\frac{dR}{dt} &= -2\alpha R - J + a_J \\ \frac{dJ}{dt} &= 2R - \alpha J + a_R\end{aligned} \tag{4}$$

(a) To get started, first use the PPLANE Java applet to plot a phase portrait when $\alpha = 0.5$, $a_R = 1$ and $a_J = 1.5$. (Hand in a print of the plot.) You should notice that the solution converges to a point that is not the origin. Do Romeo and Juliet live happily ever after in this case?

(b) Find the formula for the equilibrium solution (R_0, J_0) as a function of a_R, a_J and α. (Do not assume any particular numerical values for α, a_R or a_J for this part.)

(c) Let $U = R - R_0$ and $V = J - J_0$. Show that when expressed in terms of U and V, the differential equations become

$$\frac{d\vec{\mathbf{y}}}{dt} = A\vec{\mathbf{y}}, \quad \text{where} \quad \vec{\mathbf{y}} = \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} R - R_0 \\ J - J_0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} -2\alpha & -1 \\ 2 & -\alpha \end{bmatrix}$$

(This means that adding the constant terms to the equation simply shifts the "origin" to a new location. This is a general result *for linear systems.*)

(d) Since the equation for $\vec{\mathbf{y}}(t)$ is the same as equation (3), we know that if $\alpha > 0$, $\vec{\mathbf{y}}(t)$ will approach $\vec{\mathbf{0}}$ asymptotically, which implies $R(t) \to R_0$ and $J(t) \to J_0$.

Let $\alpha = \frac{1}{2}$, and find all the possible values of a_R and a_J for which Romeo and Juliet will approach a state of mutual attraction (i.e. for which $R(t)$ and $J(t)$ will *both* approach positive values as $t \to \infty$). Show this region by shading it in the (a_R, a_J) plane.