Math 312 - Applied Math: Social Sciences **Spring 2006**

Homework 7

Due *Friday, April 14*

1. Let A be an $m \times m$ matrix, and let a_{ij} be the element of A in row i and column j.

   (a) Suppose there is an i such that every entry in the ith row of A is zero except a_{ii}. Show that A is reducible; give a permutation matrix that transforms A into the reduced form. Also illustrate your answer with a 3×3 example.

   (b) Suppose there are indices i and j such that every entry in rows i and j is zero except a_{ii}, a_{ij}, a_{jj} and a_{ji}. Show that A is reducible; give a permutation matrix that transforms A into the reduced form. Also illustrate your answer with a 4×4 example.

   (c) Generalize parts (a) and (b). (Say something like "If there is a subset of the rows such that...")

2. If a nonnegative matrix is reducible, can it be primitive? If it is primitive, can it be reducible? Explain your answers.

3. For each of the following matrices, determine whether or not it is reducible and whether or not it is primitive. If it is reducible, find the permutation matrix P that transforms the matrix into its reduced form. If it is primitive, find the lowest integer k such that the kth power of the matrix is positive.

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 1 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0.5 & 0.5 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0.25 & 1 & 1 & 0.25 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 2 & 1/4 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1/4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

4. Draw the graph Γ associated with each of the matrices in problem 3. If the matrix is reducible, indicate in your drawing the subset of vertices that are "decoupled" from the others.

5. A certain species of animal has a life span of four years. During the first and second years, they do not reproduce. In the third and fourth years, each female has an average of $b > 0$ female offspring. (We will only keep track of the sizes of the female population.) Since b represents an average, it is not necessarily an integer. Each year, only half the animals survive to become one year older. Let $\vec{\mathbf{x}}(n) \in \mathbb{R}^4$ be the population at time n (e.g. $x_1(n)$ is the size of the population in their first year, etc.)

   (a) Find the matrix A for which the population grows according to $\vec{\mathbf{x}}(n+1) = A\vec{\mathbf{x}}(n)$.

   (b) Draw the graph $\Gamma(A)$.

   (c) Show that A is primitive.

   (d) Let $b = 4$. The Perron-Frobenius Theorem guarantees that A must have a real, positive, largest eigenvalue. This gives the long-term growth factor for the size of the total population. Find this value, and find the long term relative distribution of the size of each generation (i.e. give the relative sizes of each generation as a percentage of the total population).

   (e) Find the *exact* value of b for which, in the long-term, the total population size neither increases nor decreases. (Hint: What must the largest eigenvalue be in this case? Determine the value of b such that this eigenvalue is, in fact, a root of the characteristic polynomial of A.)