(3) In the process of creating the map, the students exercise in a systematic manner problems in the whole domain and get an overview of all the material that they have learned. Thus the unit can also serve as part of a review that teachers normally perform at the end of a course.

Considering the fact that the time spent by students in the experimental and the comparison groups was about the same, it is recommended to adopt the integrative approach that can lead to considerable gains in learning with relatively little investment of time.

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A ball rolling on a freely spinning turntable

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(Received 12 December 1996; accepted 22 January 1997)

We consider a ball rolling on a freely spinning turntable. We show that the path of the ball (in an inertial frame) is a conic section. This generalizes the well-known problem of a ball rolling on a turntable that is spinning with constant angular velocity, for which the path is a circle.

I. INTRODUCTION

In this article we consider a variation of the intriguing problem of a ball rolling on a turntable. (See Fig. 1.) We show that the path of a ball rolling on a *freely spinning* turntable is a conic section.

When the turntable rotates at a constant angular velocity (rather than spinning freely), it is surprisingly easy to show that the path of the ball is a circle. Discussions of this problem have appeared several times in this journal.[1–7] Related problems can be found in several texts on classical mechanics; see, for example, Routh,[8] Gray,[9] or Milne.[10] The text by Milne contains a rich collection of problems related to rolling spheres[10] (pp. 351–365).

The problem is an example of a nonholonomic system, the theory of which can be found in the monograph of Neimark and Fufaev.[11] Moreover, this system has several symmetries. In a holonomic system, such symmetries would result in conserved quantities by Noether's theorem. Symmetries in nonholonomic systems, however, do not necessarily lead to conserved quantities. A discussion of symmetries and conserved quantities in nonholonomic systems can be found in the text by Arnold *et al.*[12] (pp. 82–84). The reduction of nonholonomic systems with symmetry is an active area of research.[13–16]

The ball on a freely spinning turntable has three conserved quantities: the total energy, the angular momentum about the z axis, and the z component of the angular momentum of the ball. These will be used to find the path of the ball.

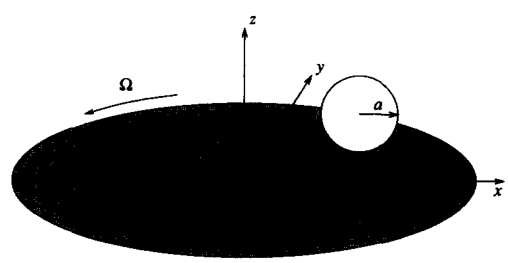

Fig. 1. A ball rolling on a freely spinning turntable. The xyz coordinate system is fixed in space, not on the turntable.

II. EQUATIONS OF MOTION

We derive the equations of motion in an inertial frame with the origin at the axis of rotation of the turntable, as shown in Fig. 1. For convenience, we list here the symbols used in the derivation:

$\mathbf{i},\mathbf{j},\mathbf{k}$	unit vectors parallel to the x, y and z directions, respectively
m,a,I_b	mass, radius, and moment of inertia of the ball
I_d	moment of inertia of the turntable about the z axis
t^*	time
$\mathbf{r}^*=x^*\mathbf{i}+y^*\mathbf{j}$	point of contact of the ball on the turntable
$\mathbf{f}^*=f_x^*\mathbf{i}+f_y^*\mathbf{j}$	reaction force acting on the ball at the point of contact
$\boldsymbol{\omega}^*=\omega_x^*\mathbf{i}+\omega_y^*\mathbf{j}+\omega_z^*\mathbf{k}$	angular velocity of the ball
$\Omega^*\mathbf{k}$	angular velocity of the turntable

The asterisks are used to indicate dimensional variables. We use a dot to indicate a derivative with respect to t^*.

We begin our derivation of the equations of motion with Newton's second law for the motion of the center of mass of the ball. The force due to gravity is balanced by the vertical force of the turntable on the ball, so the net force on the ball is $\mathbf{f}^*$. Thus

$$m\ddot{\mathbf{r}}^*=\mathbf{f}^*. \tag{1}$$

Now we consider the moment balance around the center of the ball:

$$I_b\dot{\boldsymbol{\omega}}^*=-a\mathbf{k}\times\mathbf{f}^*. \tag{2}$$

The only forces acting on the ball are in the xy plane at the point of contact, so the moment about the center of the ball has no z component. Thus ω_z^*, the z component of the ball's angular velocity, is constant.

We are assuming that the ball rolls on the turntable without slipping. This means that the velocity of the point on the ball that is in contact with the turntable must equal the velocity of the point on the turntable that is in contact with the ball. Thus

$$\dot{\mathbf{r}}^*+a\mathbf{k}\times\boldsymbol{\omega}^*=\Omega^*\mathbf{k}\times\mathbf{r}^*. \tag{3}$$

The last fundamental equation comes from the moment balance of the turntable:

$$I_d\dot{\Omega}^*\mathbf{k}=-\mathbf{r}^*\times\mathbf{f}^*. \tag{4}$$

In subsequent calculations, it proves convenient to use nondimensional variables. Let $W=\Omega^*(0)$. We make the following definitions:

$$t=Wt^*,\quad \mathbf{r}(t)=\frac{\mathbf{r}^*(t^*)}{a},\quad \mathbf{f}(t)=\frac{\mathbf{f}^*(t^*)}{maW^2},$$

$$\boldsymbol{\omega}(t)=\frac{\boldsymbol{\omega}^*(t^*)}{W},\quad \Omega(t)=\frac{\Omega^*(t^*)}{W}.$$

The nondimensional versions of Eqs. (1)–(4) are

$$\mathbf{r}''=\mathbf{f}, \tag{5}$$

$$\beta\boldsymbol{\omega}'=-\mathbf{k}\times\mathbf{f}, \tag{6}$$

$$\mathbf{r}'+\mathbf{k}\times\boldsymbol{\omega}=\Omega\mathbf{k}\times\mathbf{r}, \tag{7}$$

and

$$\delta\Omega'\mathbf{k}=-\mathbf{r}\times\mathbf{f}, \tag{8}$$

where $\beta=I_b/ma^2$, $\delta=I_d/ma^2$, and a prime (′) denotes a derivative with respect to t.

Our goal now is to eliminate $\mathbf{f}$ and $\boldsymbol{\omega}$, and obtain a set of equations for $\mathbf{r}$ and Ω only. By eliminating $\mathbf{f}$ from (5) and (6) we obtain

$$\mathbf{r}''=\beta\mathbf{k}\times\boldsymbol{\omega}'. \tag{9}$$

Integrate this from 0 to t, and use the no-slip constraint (7) to eliminate $\mathbf{k}\times\boldsymbol{\omega}$. The result is

$$\mathbf{r}'=\alpha\Omega\mathbf{k}\times\mathbf{r}+\mathbf{c}, \tag{10}$$

where $\alpha=\beta/(\beta+1)$ and $\mathbf{c}=\mathbf{r}'(0)-\alpha\mathbf{k}\times\mathbf{r}(0)$.

Suppose now that the turntable spins at a constant rate. By our choice of variables, this means that $\Omega(t)=1$ for all t. In this case, Eq. (10) is the equation for a simple harmonic oscillator written as a first-order system. Thus we have the surprising result that the path of the ball is a circle. The center of the path is $\mathbf{r}_c=\mathbf{k}\times\mathbf{c}/\alpha=\mathbf{k}\times\mathbf{r}'(0)/\alpha+\mathbf{r}(0)$. Let $\mathbf{r}(0)=X\mathbf{i}+Y\mathbf{j}$ and $\mathbf{r}'(0)=U\mathbf{i}+V\mathbf{j}$. Then the equation for the circle is

$$(x-X+V/\alpha)^2+(y-Y-U/\alpha)^2=(U^2+V^2)/\alpha^2. \tag{11}$$

In the problem we are considering, however, Ω is not constant. We find the equation for Ω by using (5), (6), and (10) to eliminate $\mathbf{f}$ from (8). The result is

$$\Omega'=\frac{-\alpha(\mathbf{r}\cdot\mathbf{c})}{\delta+\alpha(\mathbf{r}\cdot\mathbf{r})}\Omega. \tag{12}$$

Notice that if $\Omega(t_0)=0$ for some t_0, then $\Omega(t)=0$ for all t. We have assumed that $\Omega(0)=1$; therefore we conclude that $\Omega(t)>0$ for all t. It is impossible for the turntable to reverse the direction of its spin.

III. SOLUTION USING CONSERVED QUANTITIES

We use the conservation of energy and angular momentum to determine the path of the ball. We find that the path of the ball is a conic section; that is, the path is given by a quadratic equation in x and y.

Define the product $\mathbf{a}\wedge\mathbf{b}$ of two vectors in the xy plane to be

$$\mathbf{a}\wedge\mathbf{b}\equiv\mathbf{k}\cdot(\mathbf{a}\times\mathbf{b}).$$

The total (nondimensional) angular momentum μ about the z axis is

$$\mu=\delta\Omega+(\mathbf{r}\wedge\mathbf{r}')+\beta\omega_z.$$

There are no external torques about the z axis of the system, so μ is constant. We showed earlier that ω_z is constant; therefore

$$L=\delta\Omega+(\mathbf{r}\wedge\mathbf{r}')$$

is also constant. By using this equation and (10) we obtain

$$\Omega=\frac{L-(\mathbf{r}\wedge\mathbf{c})}{\delta+\alpha(\mathbf{r}\cdot\mathbf{r})}. \tag{13}$$

No work is done on the system, and the internal constraint force $\mathbf{f}$ does no work. Therefore the kinetic energy $K=\frac{1}{2}\delta\Omega^2+\frac{1}{2}\beta(\boldsymbol{\omega}\cdot\boldsymbol{\omega})+\frac{1}{2}(\mathbf{r}'\cdot\mathbf{r}')$ is also conserved. Let $\boldsymbol{\omega}_p=\omega_x\mathbf{i}+\omega_y\mathbf{j}$, so $\boldsymbol{\omega}=\boldsymbol{\omega}_p+\omega_z\mathbf{k}$. Because ω_z is constant,

$$T=\tfrac{1}{2}\delta\Omega^2+\tfrac{1}{2}\beta(\boldsymbol{\omega}_p\cdot\boldsymbol{\omega}_p)+\tfrac{1}{2}(\mathbf{r}'\cdot\mathbf{r}')$$

is also constant.

The no-slip constraint (7) can be used to express $\boldsymbol{\omega}_p$ in terms of $\mathbf{r}$ and $\dot{\mathbf{r}}$. From (7) and (10) we obtain

$$\boldsymbol{\omega}_p=(1-\alpha)\Omega\mathbf{r}+\mathbf{k}\times\mathbf{c}. \tag{14}$$

Now use (10), (13), and (14) to eliminate $\boldsymbol{\omega}_p$, $\mathbf{r}'$, and Ω from the expression for the kinetic energy. We arrive at the following equation satisfied by $\mathbf{r}=x\mathbf{i}+y\mathbf{j}$:

$$Ax^2+Bxy+Cy^2+Dx+Ey=F, \tag{15}$$

where

$$A=2\alpha T-(\beta c_1^2+(\beta+1)c_2^2),\quad B=2c_1c_2,$$
$$C=2\alpha T-((\beta+1)c_1^2+\beta c_2^2),\quad D=2Lc_2,$$
$$E=-2Lc_1,\quad F=L^2-2\delta T+\delta(c_1^2+c_2^2)(\beta+1),$$

and c_1 and c_2 are the components of $\mathbf{c}$. In terms of the initial conditions $x(0)=X$, $y(0)=Y$, $x'(0)=U$, $y'(0)=V$, and $\Omega(0)=1$, we have

$$c_1=U+\alpha Y,\quad c_2=V-\alpha X,$$
$$L=\delta+(XV-YU),$$
$$T=\tfrac{1}{2}\delta+\tfrac{1}{2}\beta(X^2+Y^2-2(XV-YU)+U^2+V^2)$$
$$+\tfrac{1}{2}(U^2+V^2).$$

Several special cases are considered in the following examples.

Example 1: We expect that when the moment of inertia of the turntable is large (i.e., when δ is large), the behavior of the system should approach that of the case where the turntable spins at a constant angular velocity. Indeed, by dividing (15) by $\alpha\delta$ and taking the limit $\delta\rightarrow\infty$, we obtain (11), the equation of the circular path of the ball. A natural question to ask is whether circular motion is possible when $\delta<\infty$. For circular motion, we must have $A=C$ and $B=0$ in Eq. (15). It follows that $c_1=c_2=0$. In this case, Eq. (15) simplifies to

$$x^2+y^2=X^2+Y^2,$$

so the ball travels in a circle around the origin. From $\mathbf{c}=0$ we can also deduce this directly from the equations of motion (10) and (12). In Eq. (12) we see that Ω' is zero, so $\Omega(t)=1$ for all t. Then Eq. (10) shows that the ball travels in a circle about the origin with speed α.

Example 2: Consider a ball initially at the origin and moving in the x direction, so $X=0$, $Y=0$, and $V=0$. The resulting equation for the path of the ball is

$$\alpha\delta x^2+(\alpha\delta-U^2)y^2-2\delta Uy=0.$$

If $U^2=\alpha\delta$, the ball travels along the parabola $y=\frac{1}{2}\sqrt{(\alpha/\delta)}x^2$. Otherwise, we can rewrite the equation as

$$\left(\frac{\alpha\delta}{\alpha\delta-U^2}\right)x^2+\left(y-\frac{\delta U}{\alpha\delta-U^2}\right)^2=\left(\frac{\delta U}{\alpha\delta-U^2}\right)^2.$$

When $U^2>\alpha\delta$, the ball rolls away from the origin along a hyperbola. If $U^2<\alpha\delta$, the ball travels along an ellipse, periodically returning to the origin.

Example 3: We now ask what happens when the ball initially has no translational velocity, so $U=V=0$. In this case, Eq. (15) simplifies to

$$\delta((x-X)^2+(y-Y)^2)+\alpha(Yx-Xy)^2=0,$$

so the ball remains spinning in place at $x=X$ and $y=Y$. This can also be deduced from the equations of motion (10) and (12), for when $c_1=\alpha Y$ and $c_2=-\alpha X$, both $\Omega'(0)$ and $\mathbf{r}'(0)$ are zero.

IV. SOME FINAL REMARKS

In Part V of their paper, Gersten, Soodak, and Tiersten[4] constructed a problem in which the paths of the ball were also conic sections. They imagined the turntable to be composed of a set of concentric rings of infinitesimal widths, with the ring of radius r rotating at the constant rate $\Omega(r)$. They showed that when $\Omega(r)=b/(\alpha r)$ (where b is a constant with the dimension of speed), the path of the ball is a conic section. The ball rolling on a freely spinning turntable provides a more realistic mechanical system for which the paths are also conic sections.

The relationship between the case where the turntable spins at a constant angular velocity and the freely spinning case is the same as that described by Kaplan.[17] He compares the system of a bead sliding in a circular wire spinning with constant angular velocity to a system in which the wire can spin freely. As he points out, there are many systems for which this type of comparison is useful.

Many of the algebraic manipulations required to derive the path of the ball given by Eq. (15) were done using the software package MAPLE. Doing these manipulations entirely by hand would have been tedious, and it is likely that the simple geometry of the result would have remained obscured. On the other hand, the simple geometry of the paths hints that there may be a simpler (or at least more geometric) derivation of these results; perhaps some reader will discover such a derivation.

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