CANARDS AND HORSESHOES IN THE FORCED VAN DER POL EQUATION

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Cartwright and Littlewood discovered "chaotic" solutions in the periodically forced van der Pol equation in the 1940's. Subsequent work by Levinson, Levi, and others has made this singularly perturbed system one of the archetypical dissipative systems with chaotic dynamics. Despite the extensive history of this system, many questions concerning its bifurcations and chaotic dynamics remain. We use a combination of analysis of the singular limit and numerical simulation to describe a horseshoe map that arises in the three-dimensional phase space. The canards that form at a "folded saddle" play a crucial role in this analysis.

1. Introduction

We study the periodically forced van der Pol oscillator[11] in the form

$$\begin{aligned} \varepsilon\dot{x} &= y + x - \tfrac{x^3}{3}, \\ \dot{y} &= -x + a\sin(2\pi\theta), \\ \dot{\theta} &= \omega, \end{aligned} \tag{1}$$

where $\theta \in \mathbb{R}/\mathbb{Z}$, and $0 < \varepsilon \ll 1$. The goal of this report is to combine an understanding of the fast/slow dynamics of this singularly perturbed system with numerical studies to show the source of a horseshoe map in the three-dimensional phase space. The geometric picture presented here complements the analysis of Cartwright and Littlewood[2,3,7,8].

By taking the limit $\varepsilon = 0$ in (1) we obtain a differential-algebraic equation on the critical manifold $C = \{(x, y, \theta) | y = x^3/3 - x\}$. By eliminating y and then rescaling the vector field by $x^2 - 1$, we obtain the *slow flow*:

$$\theta' = \omega(x^2 - 1), \qquad x' = -x + a\sin(2\pi\theta). \tag{2}$$

If we rescale time as $\tau = \varepsilon t$ in (1), and then take the limit $\varepsilon = 0$, we

obtain the equation for the *fast dynamics*

$$x' = y + x - x^3/3, \tag{3}$$

where y is treated as a parameter, and the fast system is independent of θ. Each point in the critical manifold C is an equilibrium point of (3). We decompose C into five subsets. $C_s^+ = \{(x, y, \theta) \in C | x > 1\}$ and $C_s^- = \{(x, y, \theta) \in C | x < -1\}$ are the subsets of C which are stable equilibria of the fast subsystem; $C_u = \{(x, y, \theta) \in C | |x| < 1\}$ is the set of unstable equilibria of (3); the two lines $S_{\pm 1} = \{(x, y, \theta) \in C | x = \pm 1\}$ are the fold lines.

When $a > 1$, the slow flow has equilibria at the points $(\theta, x) = (\pm \sin^{-1}(1/a)/2\pi, \pm 1)$. Two of these equilibria are saddle points. The other two equilibria are stable nodes if $1 < a < \sqrt{1 + 1/(16\pi\omega)^2}$ and stable foci if $a > \sqrt{1 + 1/(16\pi\omega)^2}$. These equilibria are called folded equilibria because they lie on the fold curves of the critical manifold; they are not true equilibria of the van der Pol system (1). Let P_s be the saddle point on S_{-1}, let γ_s be the stable manifold of P_s in C_s^-, and let γ_c be the segment of the invariant manifold of P_s in C_u that is the smooth continuation of γ_s from P_s to the first intersection with S_{-1}. We phrase the definition this way because, while in the rescaled slow subsystem (2) this would be a stable manifold of P_s, we note that the direction of the flow of (2) on C_u is the reverse of the actual flow of (1). Trajectories of (1) close to γ_c flow *away* from the saddle, and γ_c acts like an unstable manifold. Finally, θ_s is the θ coordinate of the saddle on S_{-1}, θ_m is the θ coordinate of the other end of γ_c, and θ_u is the θ coordinate of the first intersection with S_{-1} of the unstable manifold of P_s that lies in C_s^-. (See Figure 1.)

Figure 2 shows an example of a periodic orbit in the three dimensional phase space of (1). The vector field of the slow flow is shown in the surface C. The periodic orbit is a numerically computed solution to (1) with $\varepsilon = 10^{-4}$. As expected, the trajectory follows the slow flow when it is near C; away from C, the trajectory is approximated by the fast subsystem.

2. The Folded Saddle and Canards

The behavior of trajectories near a folded saddle was analyzed by Benoit[1] (see also Mischenko, et al[9] and Szmolyan and Wechselberger[10]). We give an informal summary of the behavior. Consider a trajectory that is near the slow manifold C_s^-. The trajectory is approximated by the slow flow until it reaches a neighborhood of the fold. At a typical fold point, the trajectory makes a transition after which it is approximated by a segment

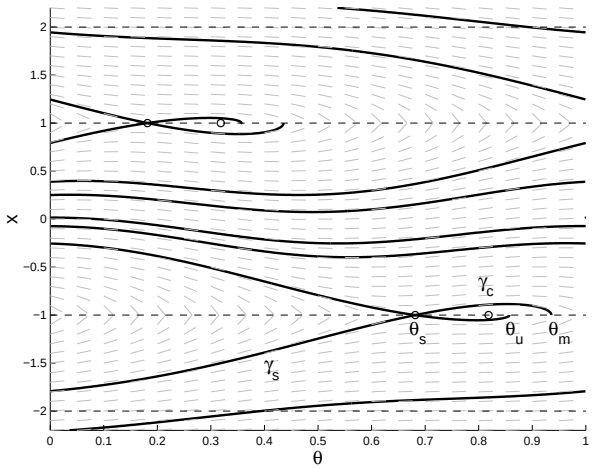

Figure 1. The phase portrait of the slow flow ($a = 1.1$, $\omega = 1.55$). The dashed lines at $x = \pm 1$ are the fold lines, and the dashed lines at $x = \pm 2$ are the projections of the fold lines in the fast direction. The small circles indicate equilibria.

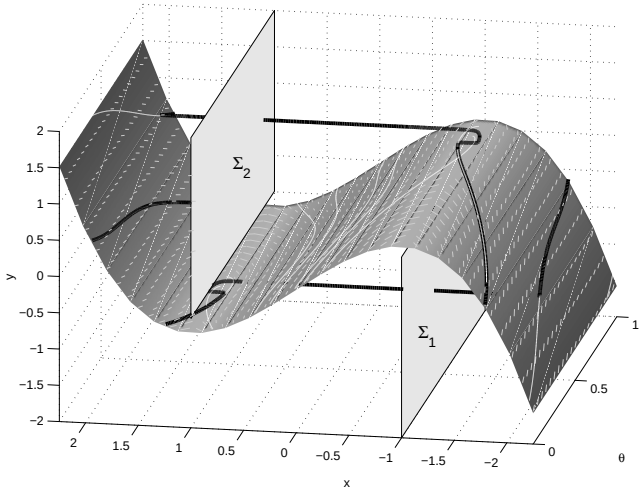

Figure 2. The gray surface is the critical manifold C. The white lines in C are the (numerically computed) invariant manifolds of the folded saddle in the slow flow. (These are the same curves plotted in black in Fig. 1.) The black curve is an unstable periodic orbit computed by AUTO, for the parameter values $\varepsilon = 0.0001$, $a \approx 1.1$, and $\omega = 1.55$.

of the fast subsystem. Informally, we call this transition a "jump at the fold". *Canards* may form near a folded saddle. These are trajectories that

do not jump at the fold, but instead remain close to the unstable sheet of the critical manifold after passing through a neighborhood of the fold. The slow segments of canards near the unstable sheet of the critical manifold will be close to γ_c. Since the trajectory is now near the unstable sheet of the slow manifold C_u, it will eventually diverge from C_u exponentially in the fast direction.

We classify the behavior of the canards by the direction in which they eventually jump off the unstable sheet. Those that jump back to the stable sheet where the trajectories began are the *jump-back* canards, and those that jump in the opposite direction (to land on a different sheet of the stable manifold) are the *jump-away* canards. The periodic orbit shown in Figure 2 contains an example of a jump-back canard. The initial conditions that give rise to the two types of canards are separated by an even smaller region in which a *maximal canard* forms. This is a canard that remains near the unstable manifold long enough to reach another fold. (In the full system, with $\varepsilon > 0$, there is not a unique maximal canard.) In this note we only consider an example where the maximal canard returns to the same fold as the one containing the folded saddle where the canard began.

3. The Poincaré Map

We define a Poincaré map Φ for (1). Let Σ_1 be the half cylinder $\{(x, y, \theta) \,|\, x = -1, y < 2/3 - \delta\}$ for some small $\delta > 0$ (see Figure 2), and let $\Sigma_2 = \{(x, y, \theta) \,|\, x = 1, y > -2/3 + \delta\}$. The system (1) is invariant under the symmetry transformation $T(x, y, \theta) = (-x, -y, \theta + 1/2)$, so rather than following the flow back to Σ_1, we use the symmetry T to map from Σ_2 back to Σ_1. For ε sufficiently small, there is an open set of parameter values (a, ω) such that the flow map from Σ_1 to Σ_2 composed with the symmetry transformation is a well defined map Φ from Σ_1 to itself.

Most trajectories beginning in Σ_1 behave as follows. There is a fast jump to a neighborhood of the critical manifold, then a slow evolution (approximated by the slow flow (2)) to a neighborhood of the fold at $x = -1$, where the trajectory will jump to Σ_2. As explained above, canards form on trajectories sufficiently close to the stable manifold of the saddle. Let $\kappa \subset \Sigma_1$ be the set of points for which canards will form during the flow from Σ_1 to Σ_2. When these trajectories finally cross Σ_2, they have either jumped from a point near γ_c directly to Σ_2 (the jump away canards), or they have jumped back to a neighborhood of C_s − and then jumped a second time, passing the fold between θ_u and θ_m (the jump back canards).

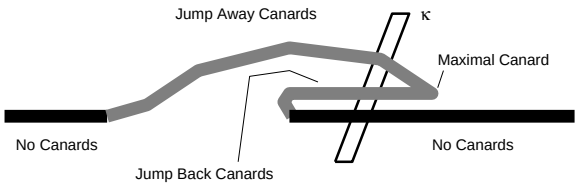

Figure 3. A qualitative sketch of the image of Σ_1 under the Poincaré map Φ. The union of the solid black and gray curves represents $\Phi(\Sigma_1)$. The white parallelogram represents a part of κ, and the gray segment represents $\Phi(\kappa)$.

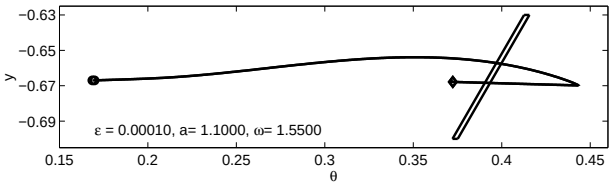

Figure 4. A Poincaré map that exhibits a horseshoe in (1). The parameter values are $\varepsilon = 10^{-4}$, $a = 1.1$ and $\omega = 1.55$.

For $0 < \varepsilon \ll 1$, κ is a thin strip, given approximately by a thickened neighborhood of the set of points in Σ_1 that project onto γ_s.

Because of the contraction onto the slow manifold, the image of Σ_1 in Σ_2 will be a thin strip. The image of $\Sigma_1 \backslash \kappa$ in Σ_2 will be a thin strip close to the line $y = 2/3$, with a gap from θ_s to θ_u. The ends of the gap are connected by the image of κ, which is also a thin strip that will be near the union of the projections onto Σ_2 of γ_u and of the fold line between θ_u and θ_m. A qualitative sketch of the Poincaré map Φ is shown in Figure 3. A horseshoe map is created when $\Phi(\kappa)$ overlaps κ, as suggested in Figure 3.

To verify this qualitative picture for a specific set of parameter values, we used the program AUTO[4] as a boundary value problem solver to compute the Poincaré map when $\varepsilon = 10^{-4}$, $a = 1.1$ and $\omega = 1.55$. The result is shown in Figure 4. The curves in Figure 4 that end with the circles and diamonds are the images under the Poincaré map Φ of the short horizontal line segments at the upper and lower ends of the thin parallelogram. The parallelogram was chosen to include a piece of the strip κ. The strong contraction on the stable slow manifold makes the resulting images indistinguishable in this plot. The combination of the contraction along the long direction of the parallelogram, the expansion along the horizontal direction (resulting from the formation of canards), and the fold that occurs at the

maximal canard are all the ingredients needed to show the existence of a horseshoe map with a hyperbolic invariant set.

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