Canards and Horseshoes in the Forced van der Pol Equation

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Collaborators

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- *John Guckenheimer*, Cornell University
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- *Ricardo Oliva*, Lawrence Berkeley National Lab
- Undergraduates (REU Summer 2002 at Cornell)
  Katy Bold, Chantal Edwards, Saby Guharay, Judith Hubbard, Chris Lipa

Brief History

$$\boxed{x'' + d(x^2-1)x' + x = a\sin(\nu\tau)}$$

- 1926 van der Pol ($a=0$): relaxation oscillations
- 1927 van der Pol and van der Mark: hysteresis and bistability
- 1940... Cartwright and Littlewood (chaos before “chaos”)
- 1949 Levinson: simplified piecewise linear model. Inspired...
- 1963 Smale: Horseshoe Map
- 1978 Levi: further simplified model, symbolic dynamics
- 1980’s Grasman (et al): Asymptotic analysis
- $\cdots$ Many more analytical and numerical studies

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New parameters: $\varepsilon = 1/d^2$, $\omega = \dfrac{\nu d}{2\pi}$

New variables: $t = \sqrt{\varepsilon}\tau$, $\theta = \omega t$, $y = \varepsilon\dot{x} + x^3/3 - x$

Then

$$\begin{aligned} \varepsilon\dot{x} &= x - \frac{1}{3}x^3 + y \\ \dot{y} &= -x + a\sin(2\pi\theta) \\ \dot{\theta} &= \omega \end{aligned}$$

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Symmetry: $x \to -x$, $y \to -y$, $\theta \to \theta + 1/2$

Slow and Fast Subsystems

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Slow and Fast Subsystems

$$\varepsilon \dot{x} = x - \frac{1}{3}x^3 + y$$
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$$\downarrow \; \varepsilon = 0$$

Slow Subsystem (DAE)

$$y = \frac{1}{3}x^3 - x$$
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$\downarrow$ Eliminate y and desingularize

$$\dot{\theta} = \omega(x^2 - 1)$$
$$\dot{x} = -x + a\sin(\theta)$$

(Time reversed for $|x| < 1$.)

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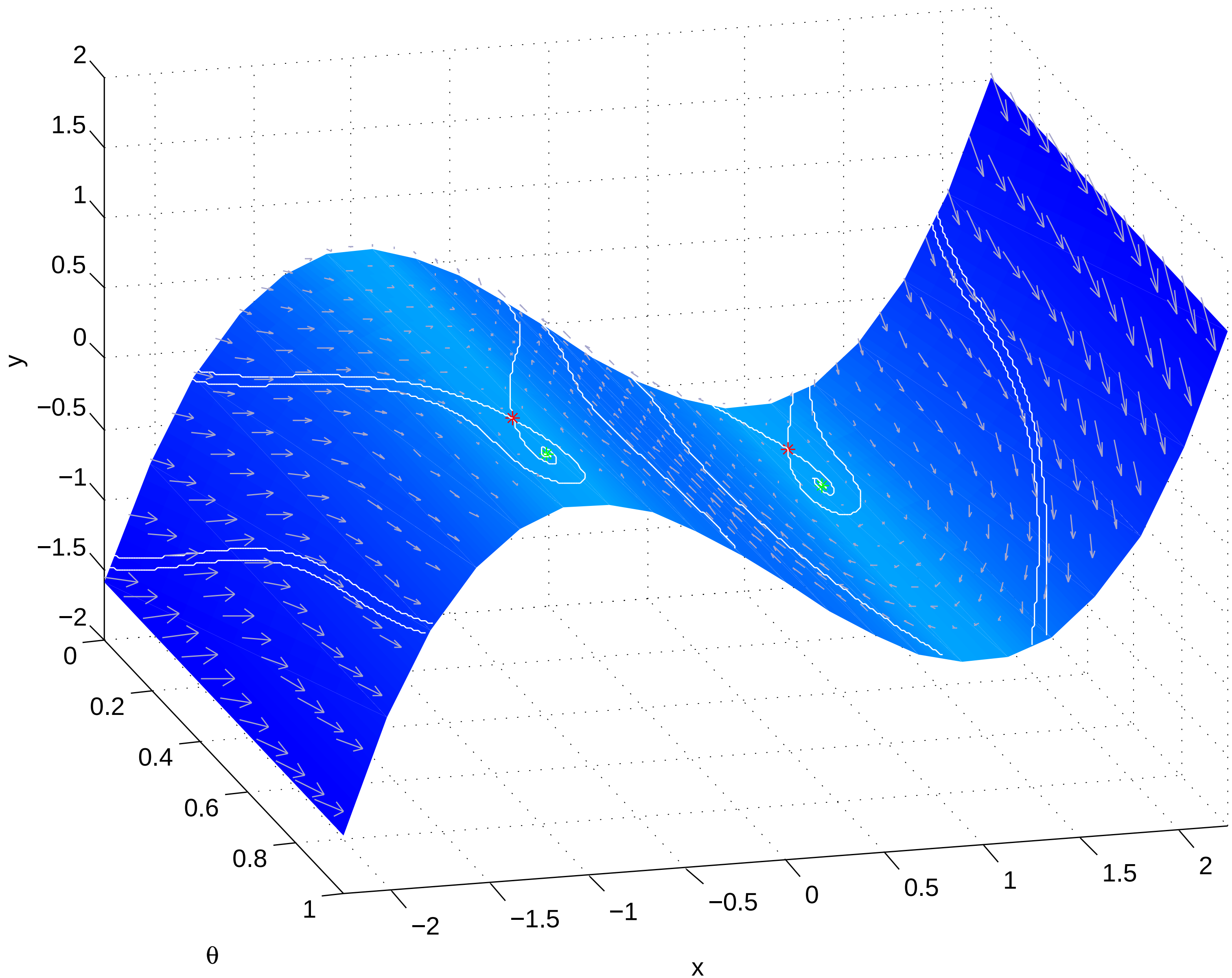
(Time reversed for $|x| < 1$.)

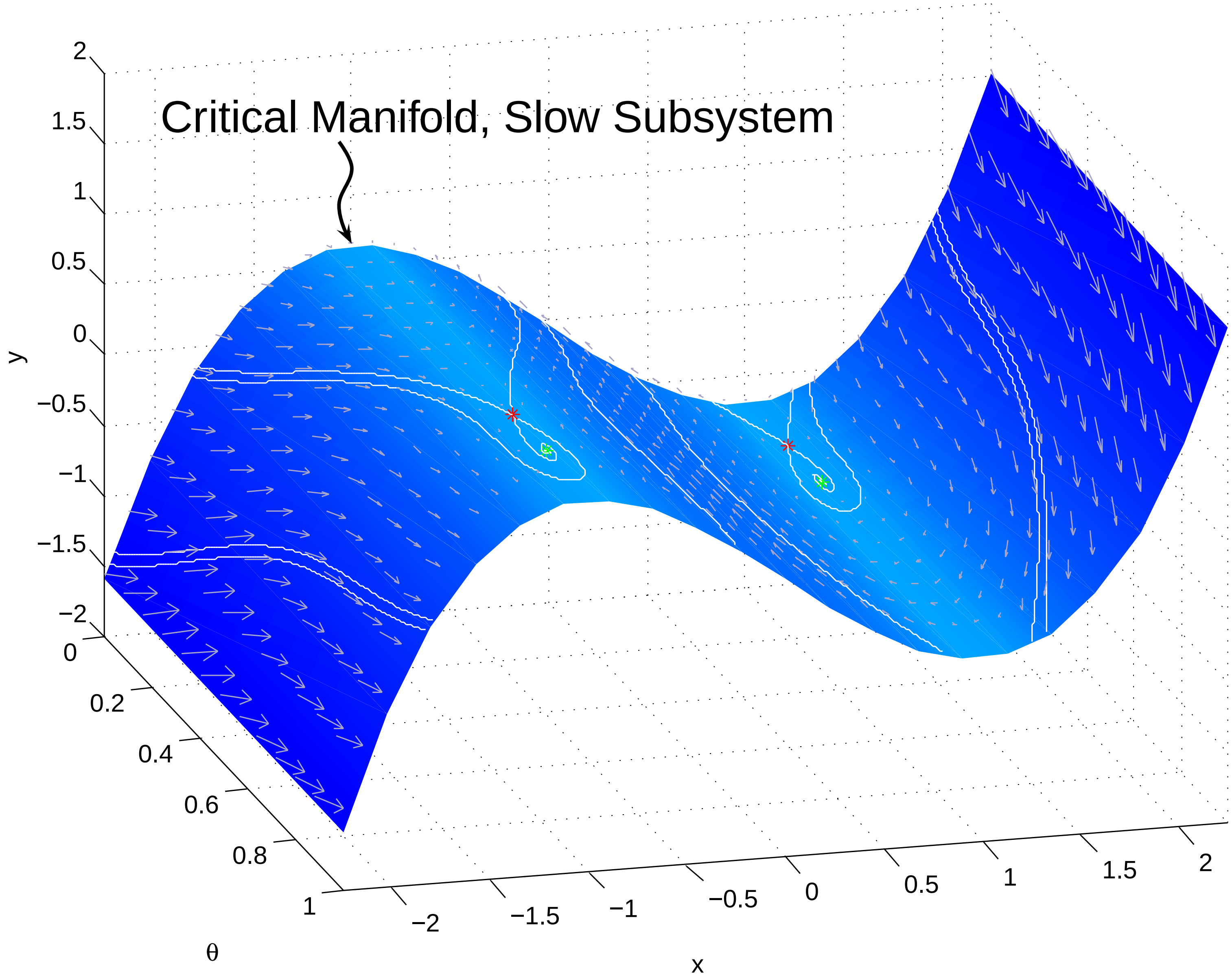
Critical Manifold

$y = \frac{1}{3}x^3 - x$

Phase Space

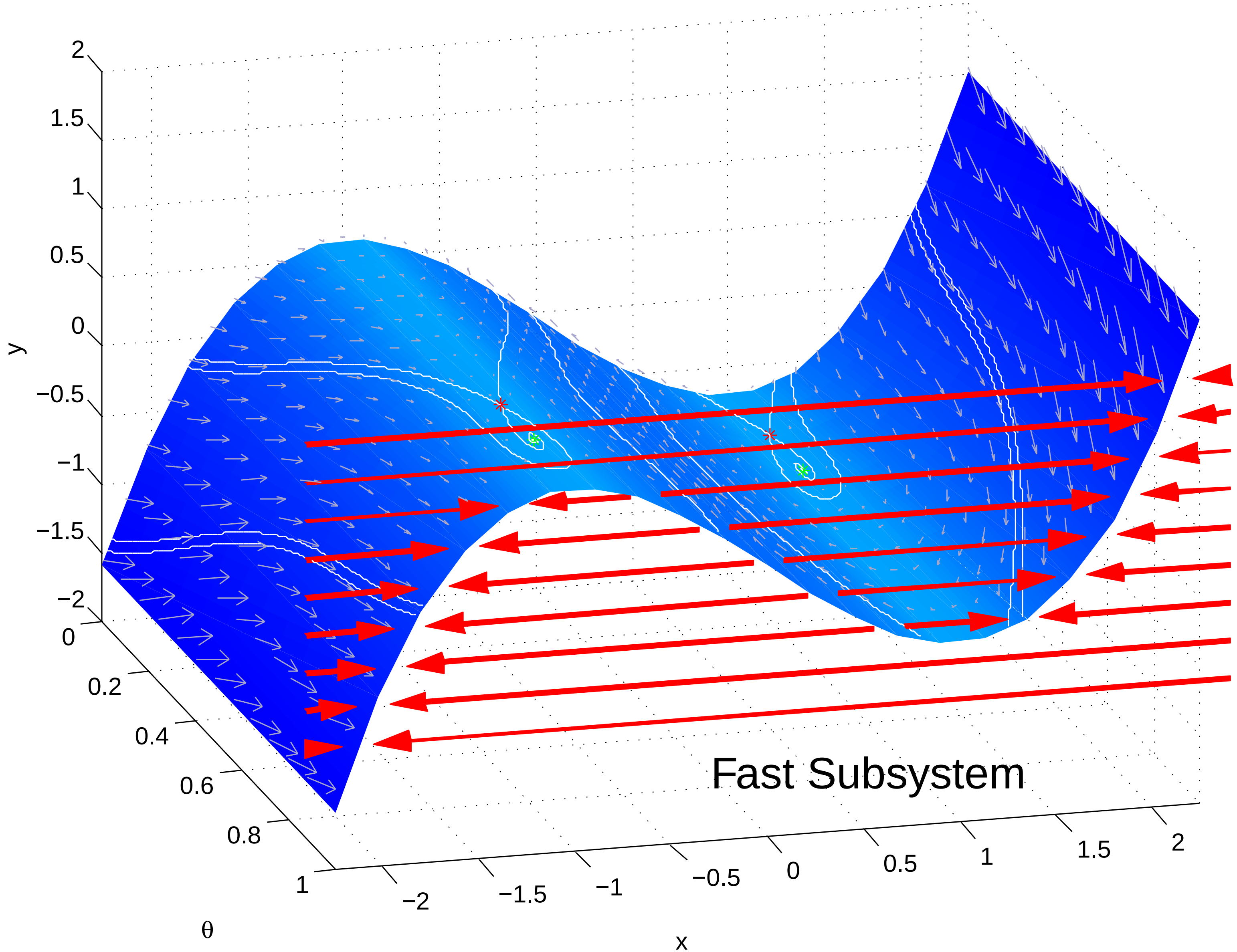


Phase Space



Critical Manifold, Slow Subsystem
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
y
0
0.2
0.4
0.6
0.8
1
θ
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
x

Phase Space



y
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0.5
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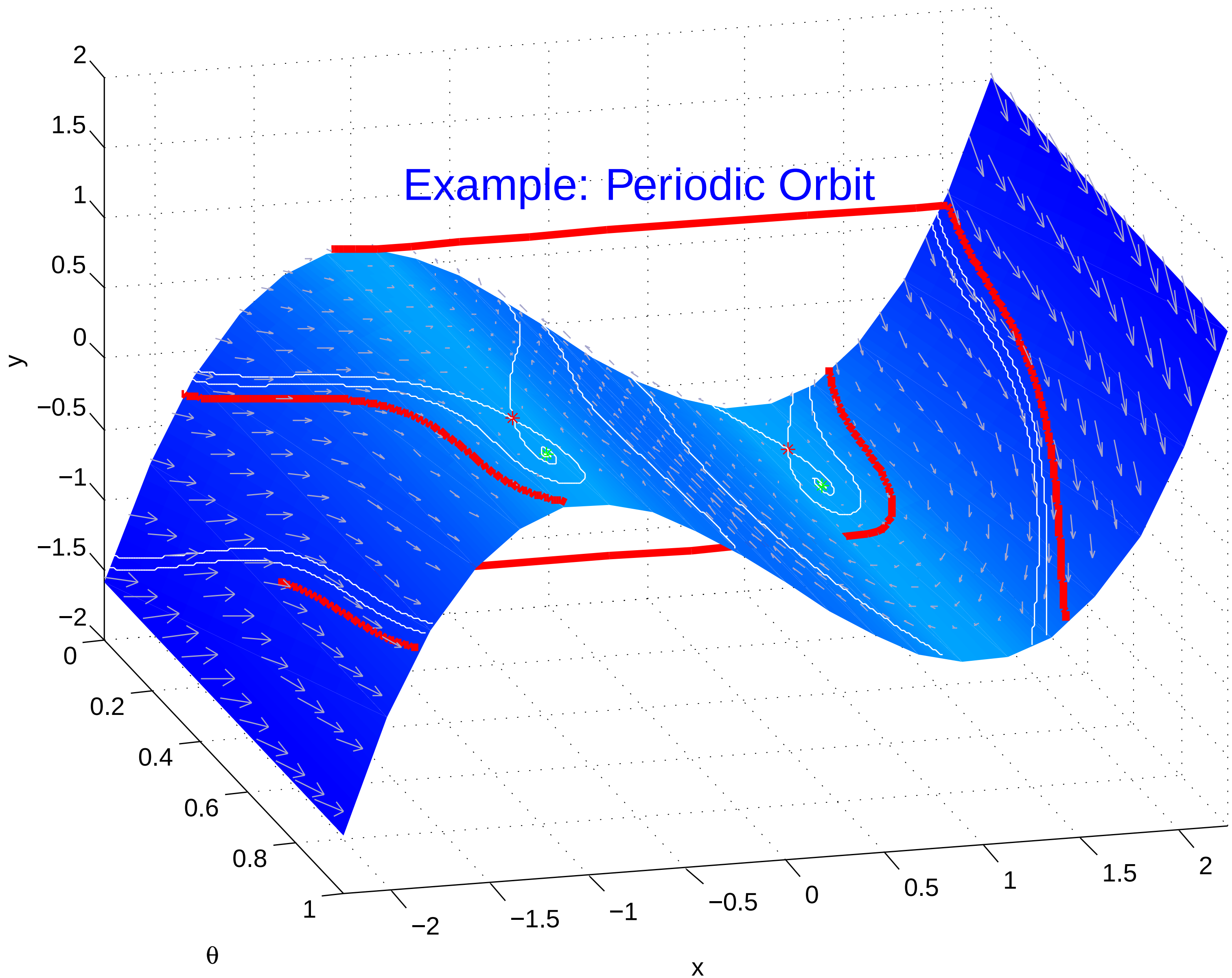
2
Fast Subsystem

Phase Space



Example: Periodic Orbit
y
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
θ
0
0.2
0.4
0.6
0.8
1
x
−2
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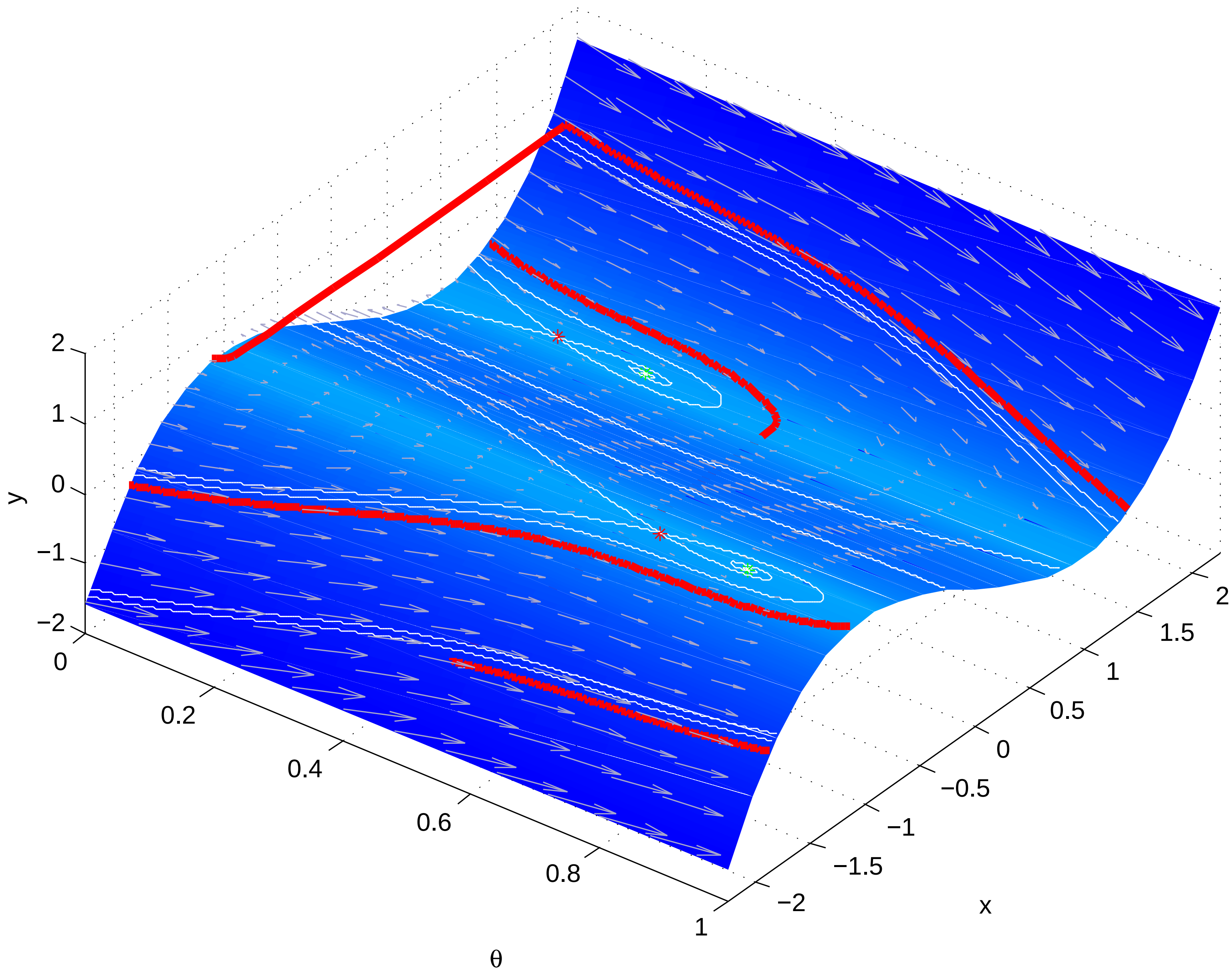
−1
−0.5
0
0.5
1
1.5
2

Phase Space



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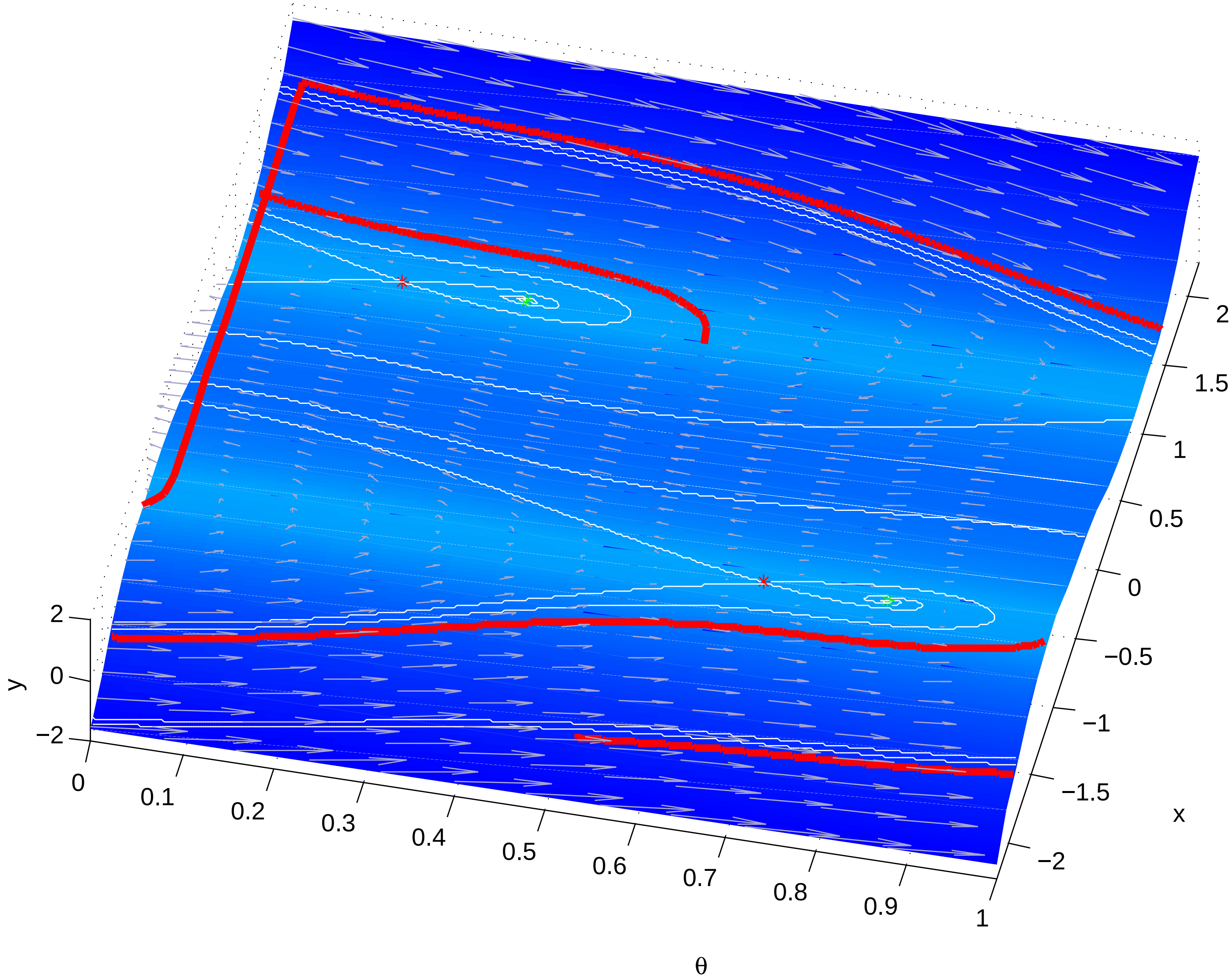
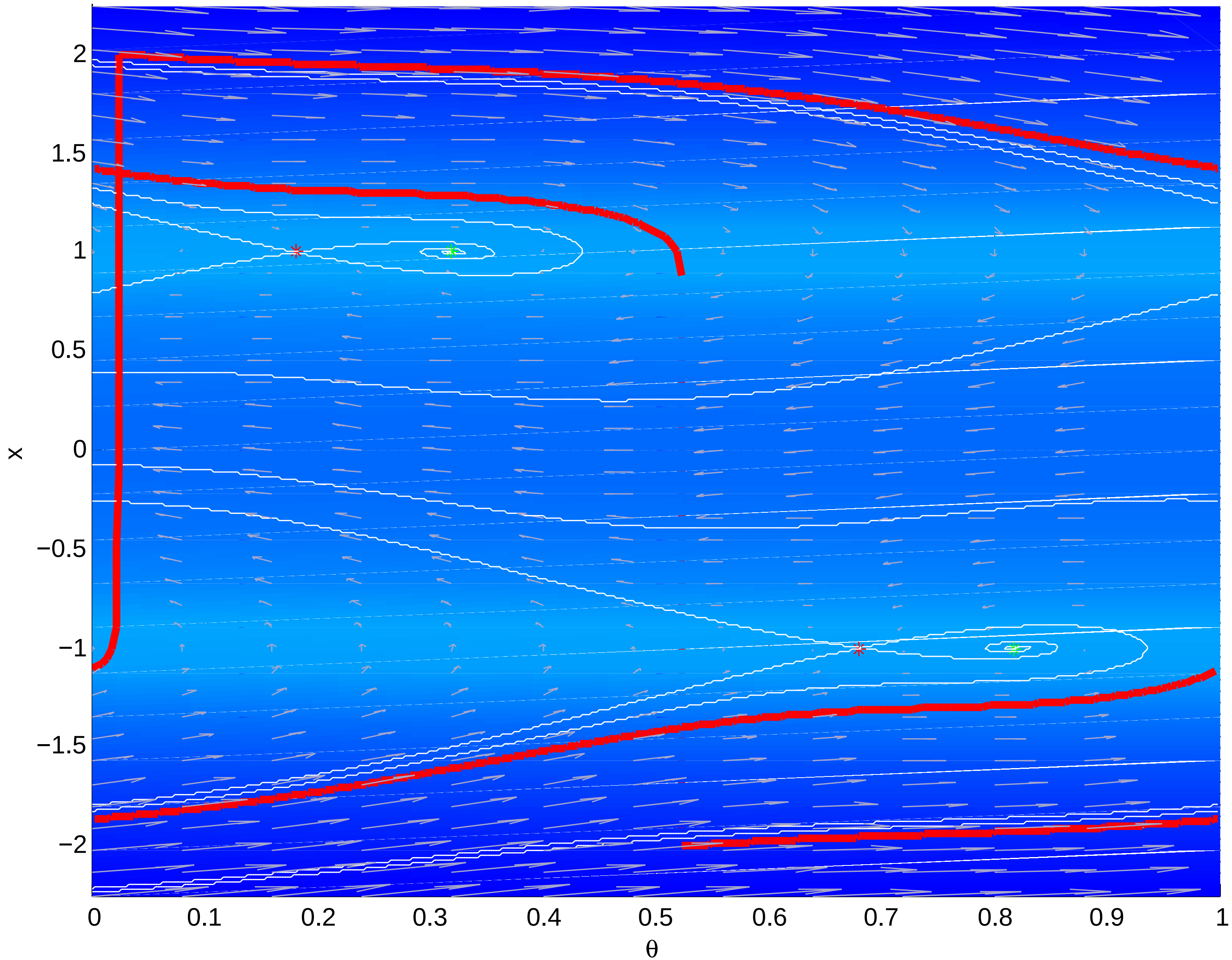
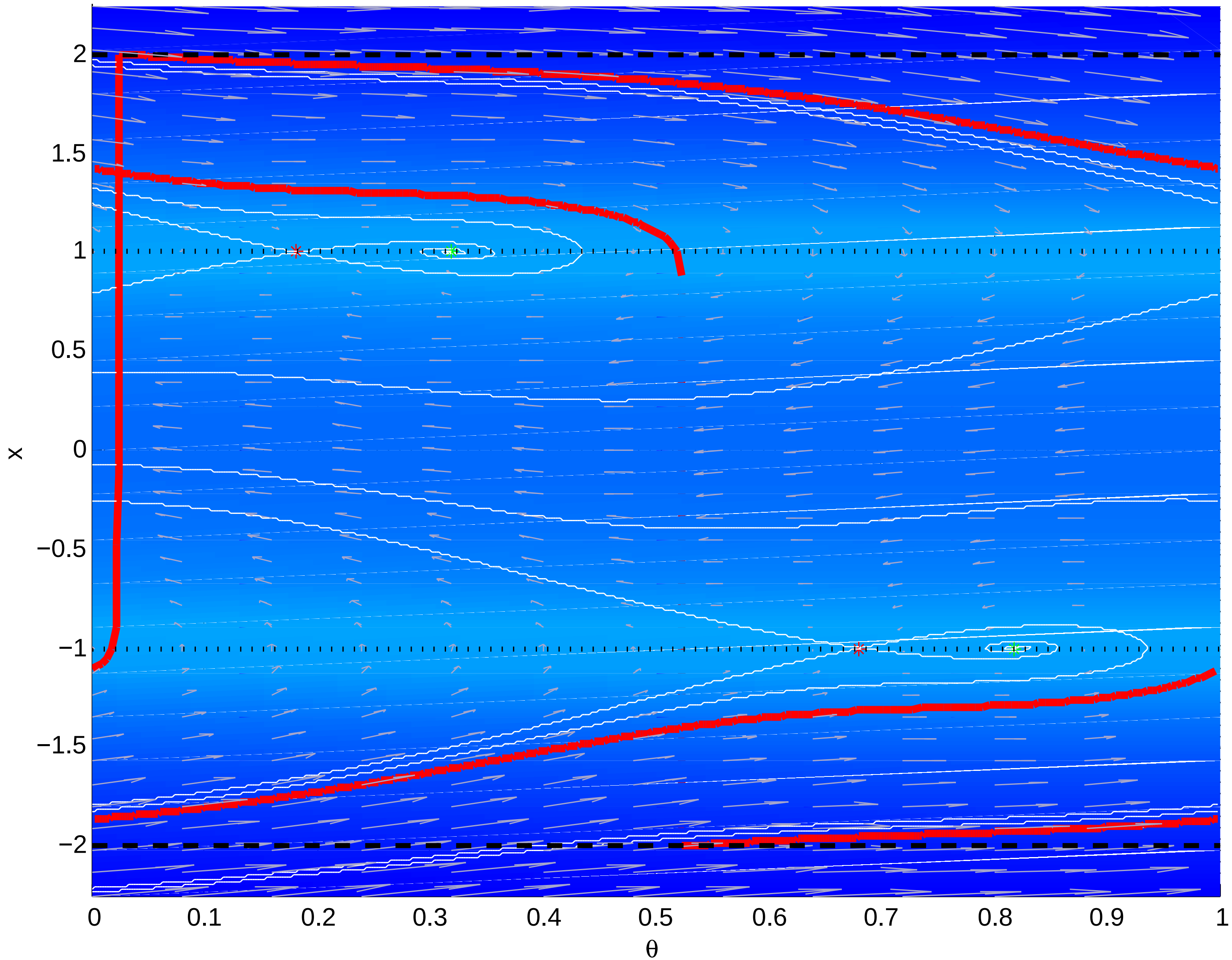


Phase Space



Phase Space



Canards at the Folded Saddle

- If $a > 1$, there is a pair of *folded equilibria* (*pseudo-singular points*) on each fold. One is a saddle, the other is either a node or a spiral.

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- *Canards* form at folded saddles.
  (Benoit 1983; Mischenko, Kolesov, Kolesov, & Rhozov 1994; Szmolyan & Wechselberger 2003)

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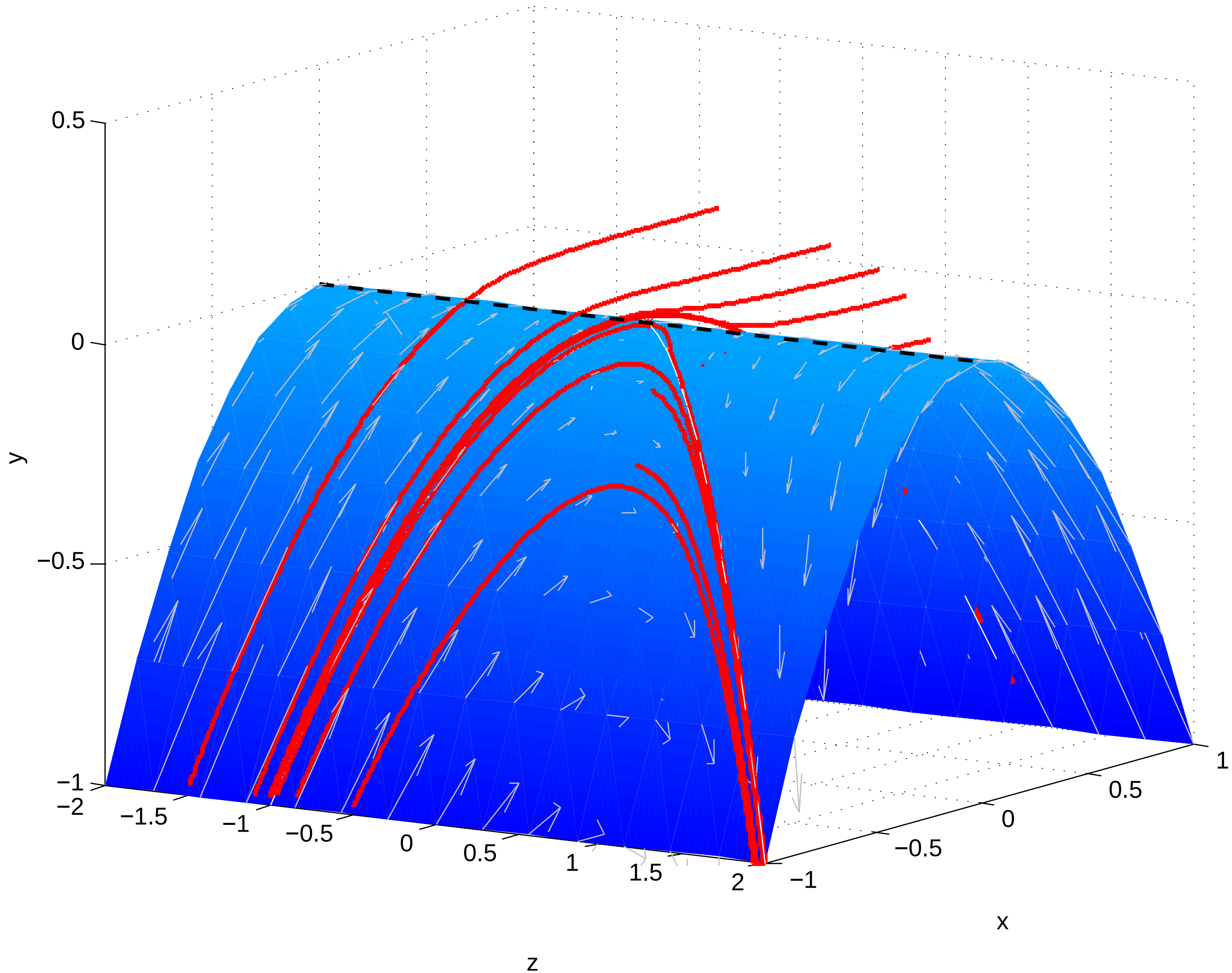
  (Benoit 1983; Mischenko, Kolesov, Kolesov, & Rhozov 1994; Szmolyan & Wechselberger 2003)
- Representative System:

$$\begin{aligned} \varepsilon\dot{x} &= y + x^2 \\ \dot{y} &= -az + bx \\ \dot{z} &= 1 \end{aligned}$$

Critical manifold is $y = -x^2$.
The origin is a folded equilibrium.

Canards at a Folded Saddle



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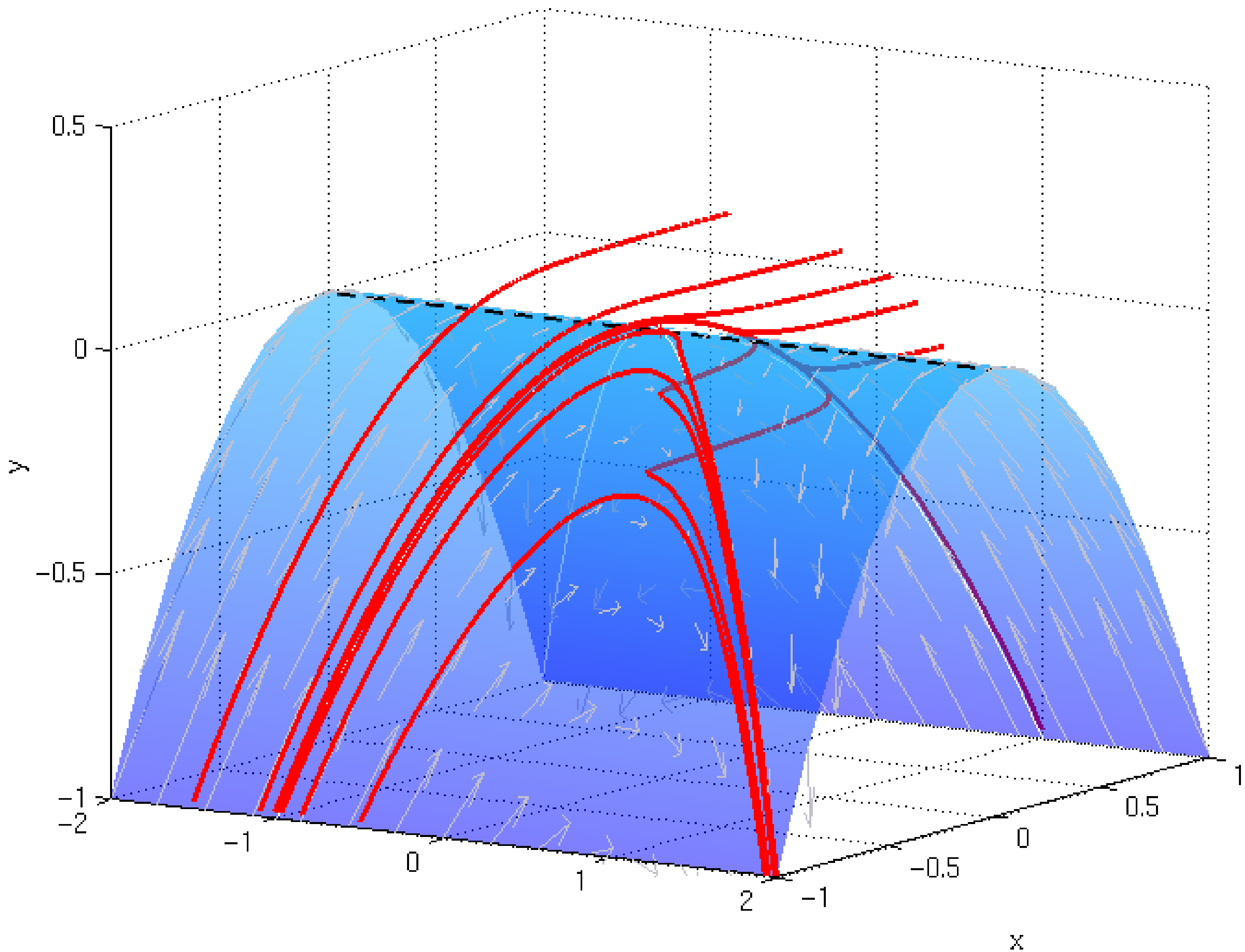

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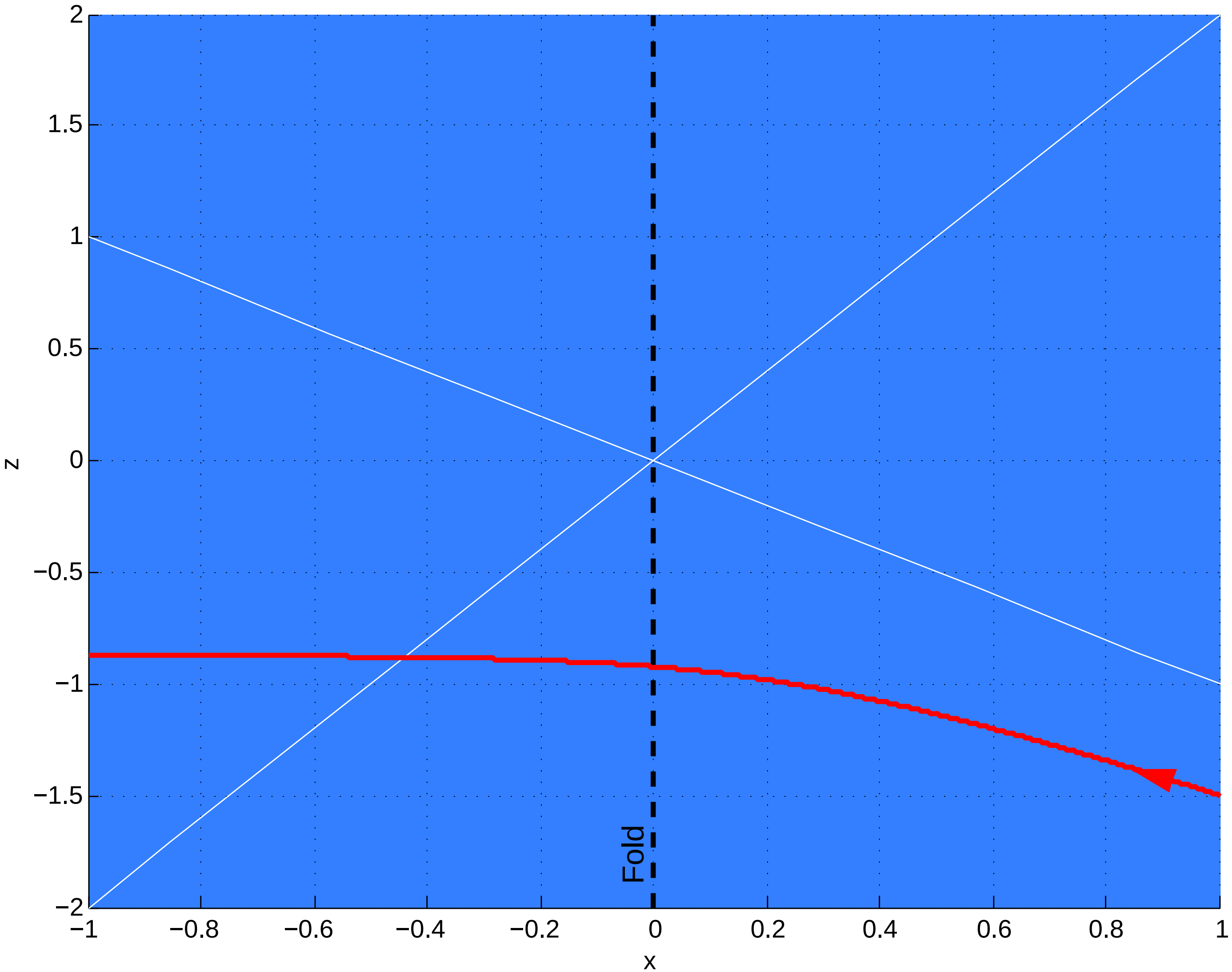

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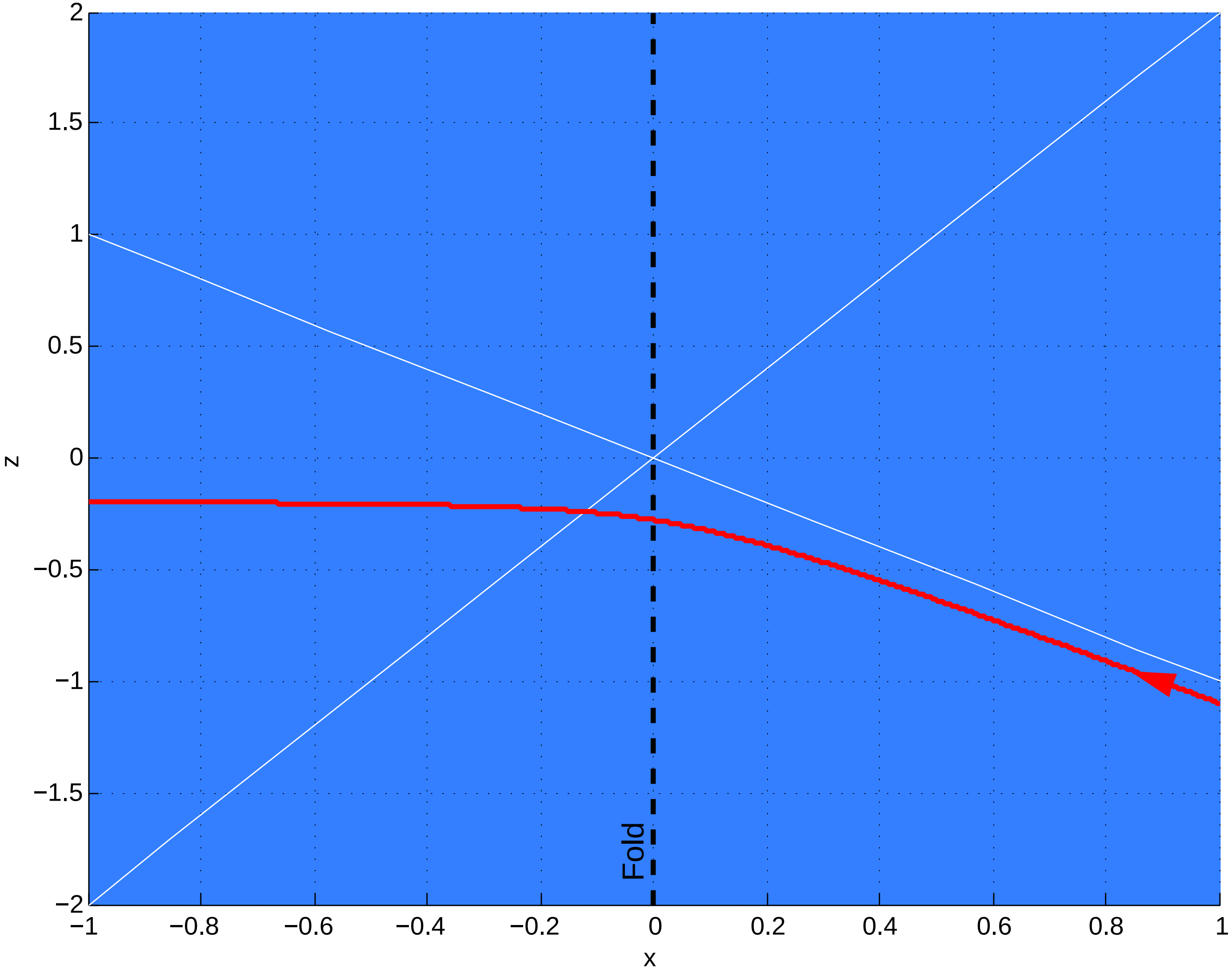

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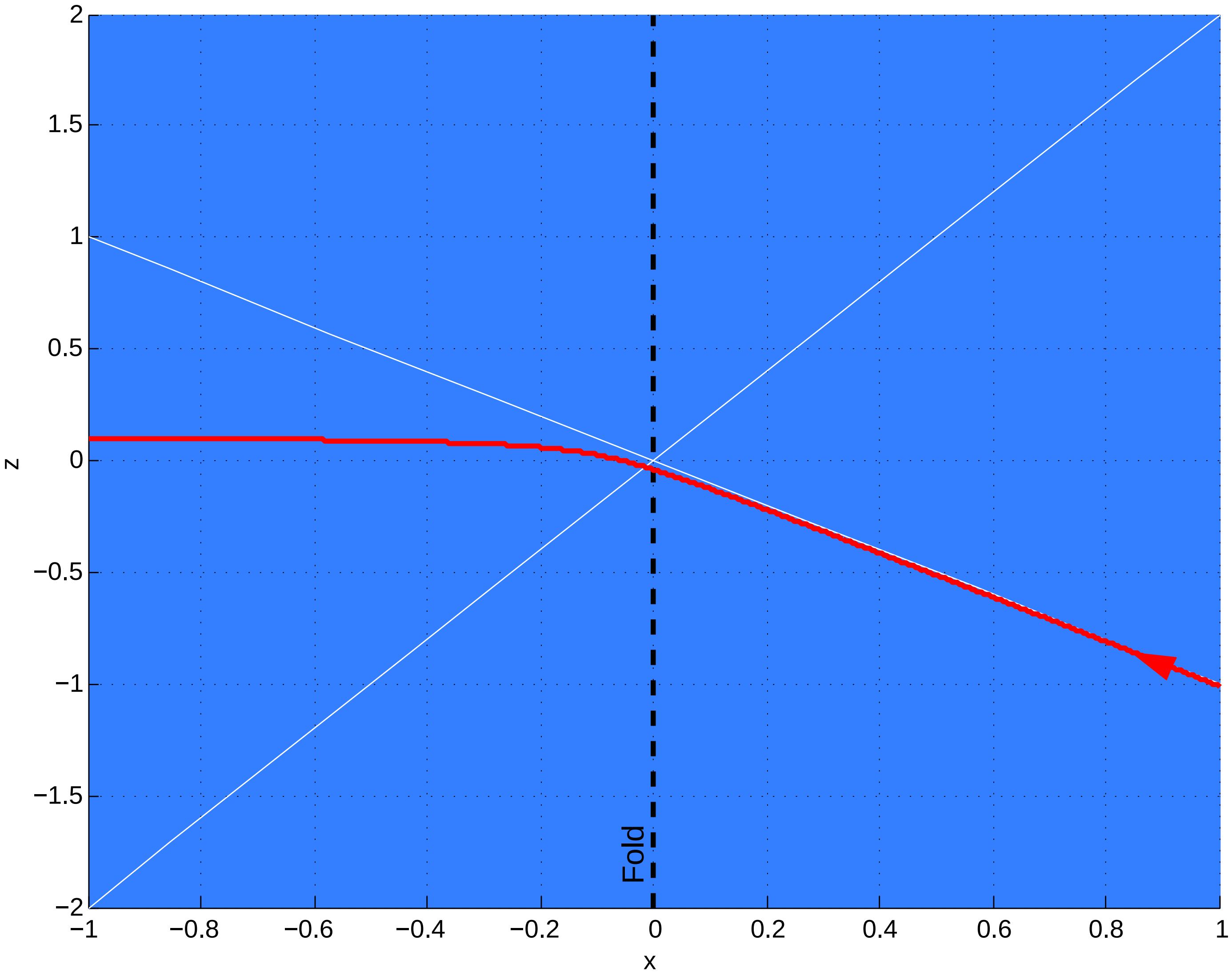

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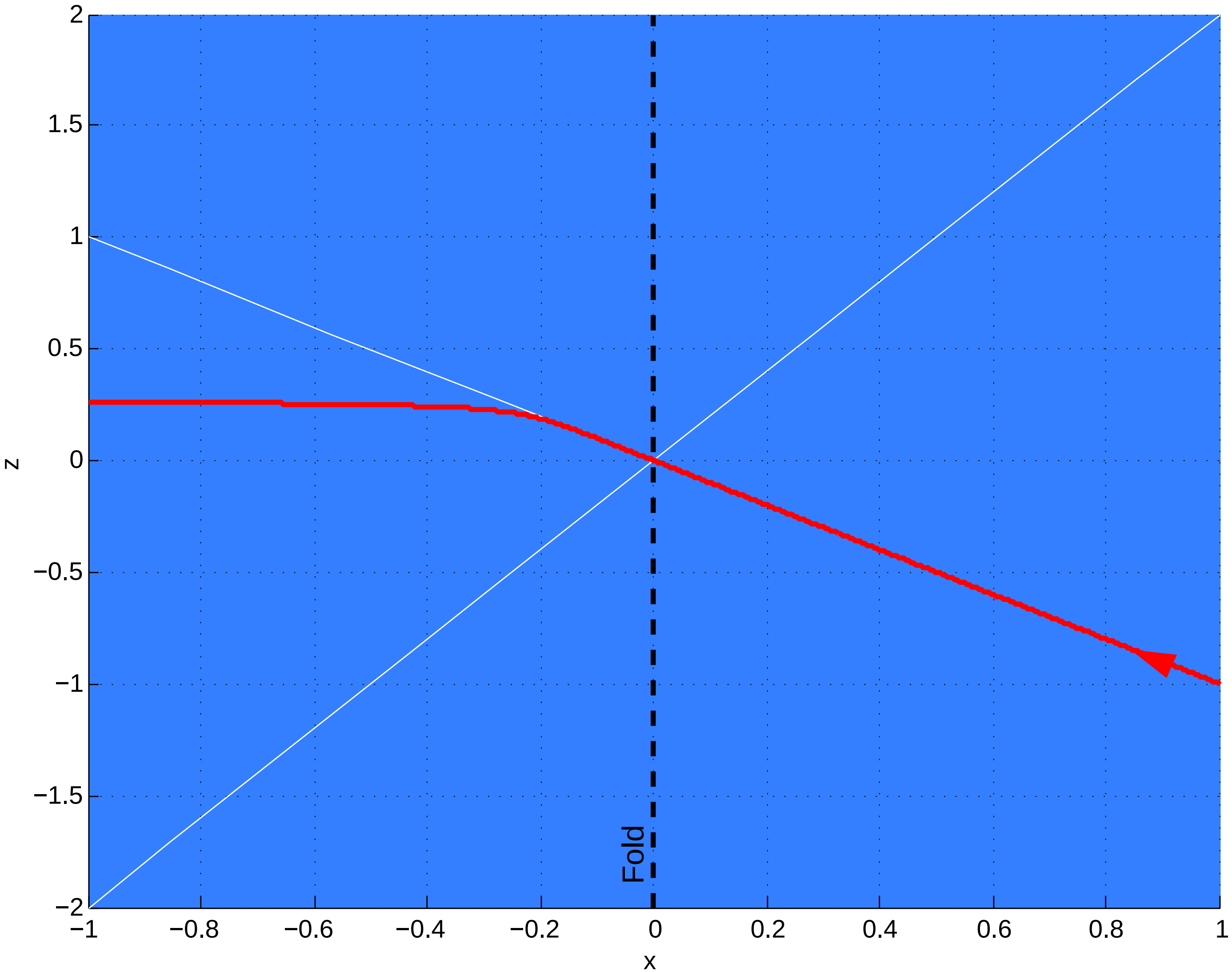

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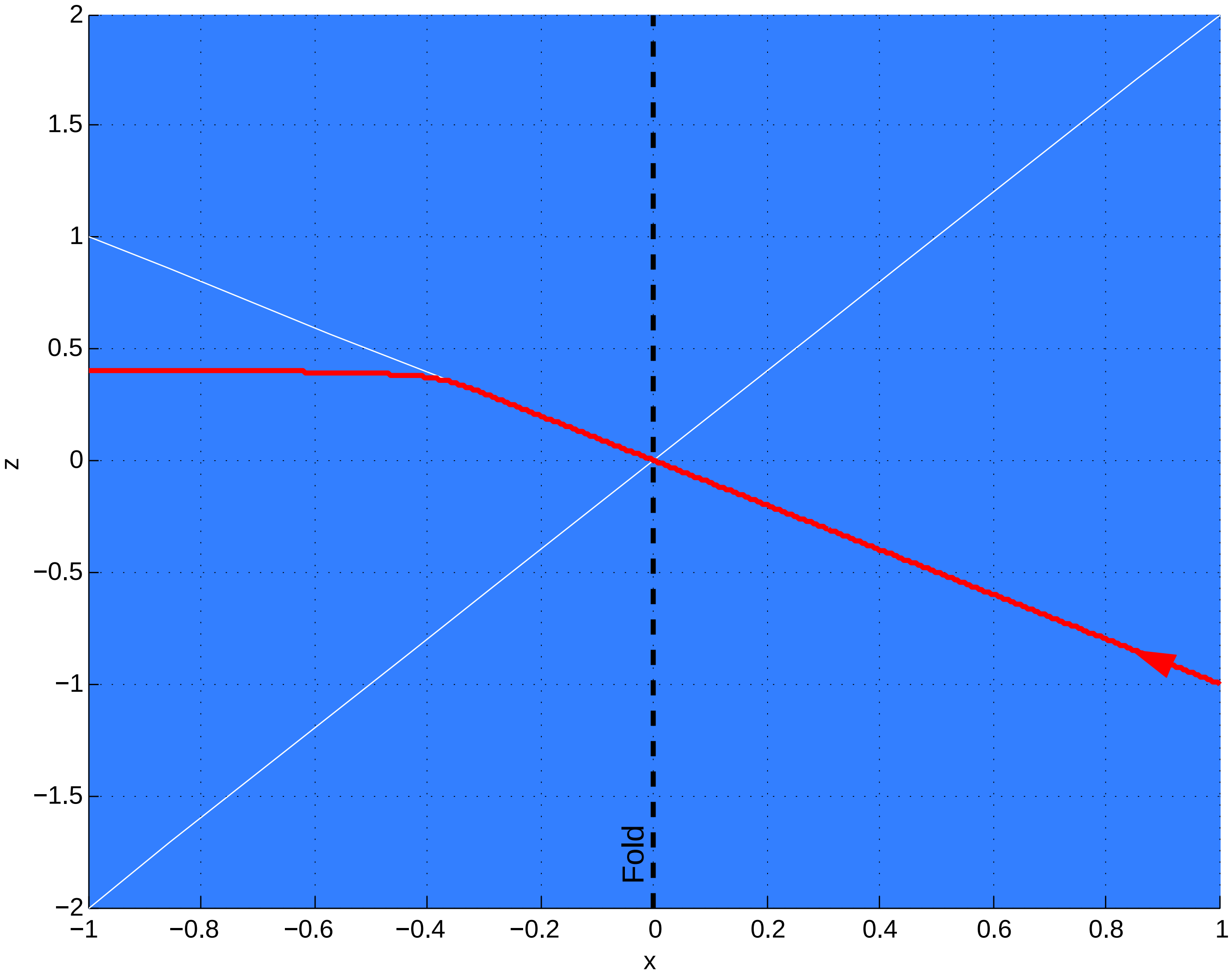

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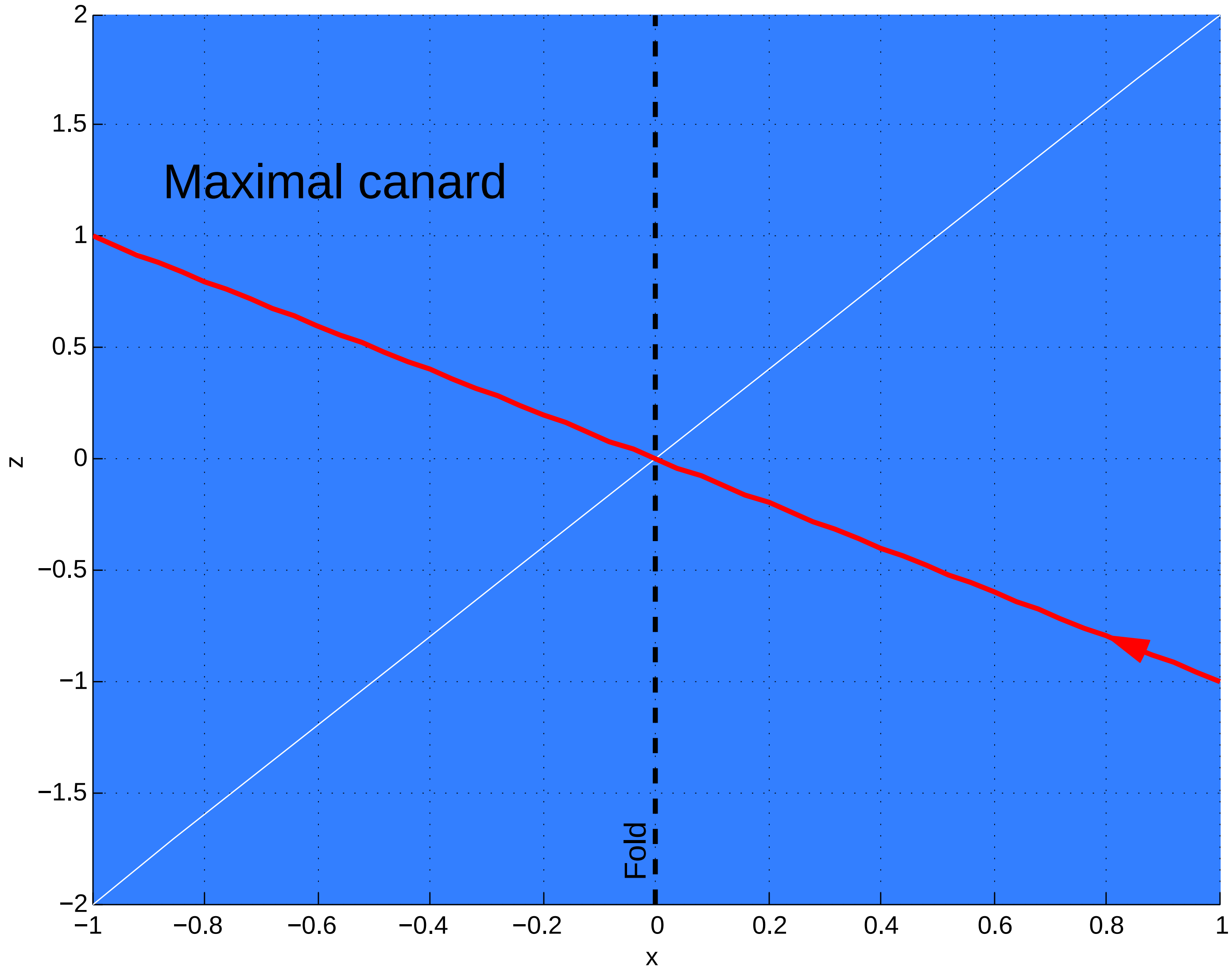

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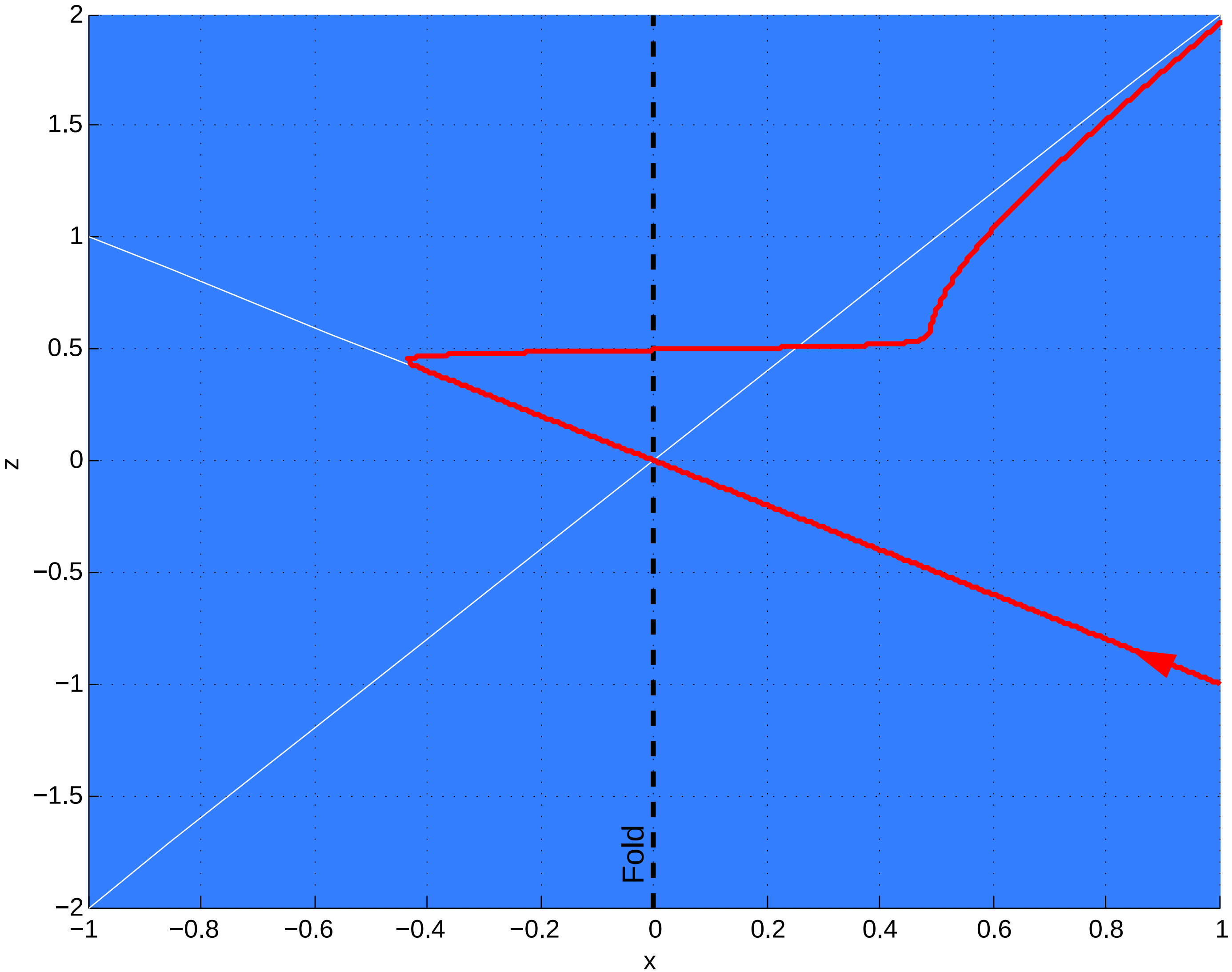

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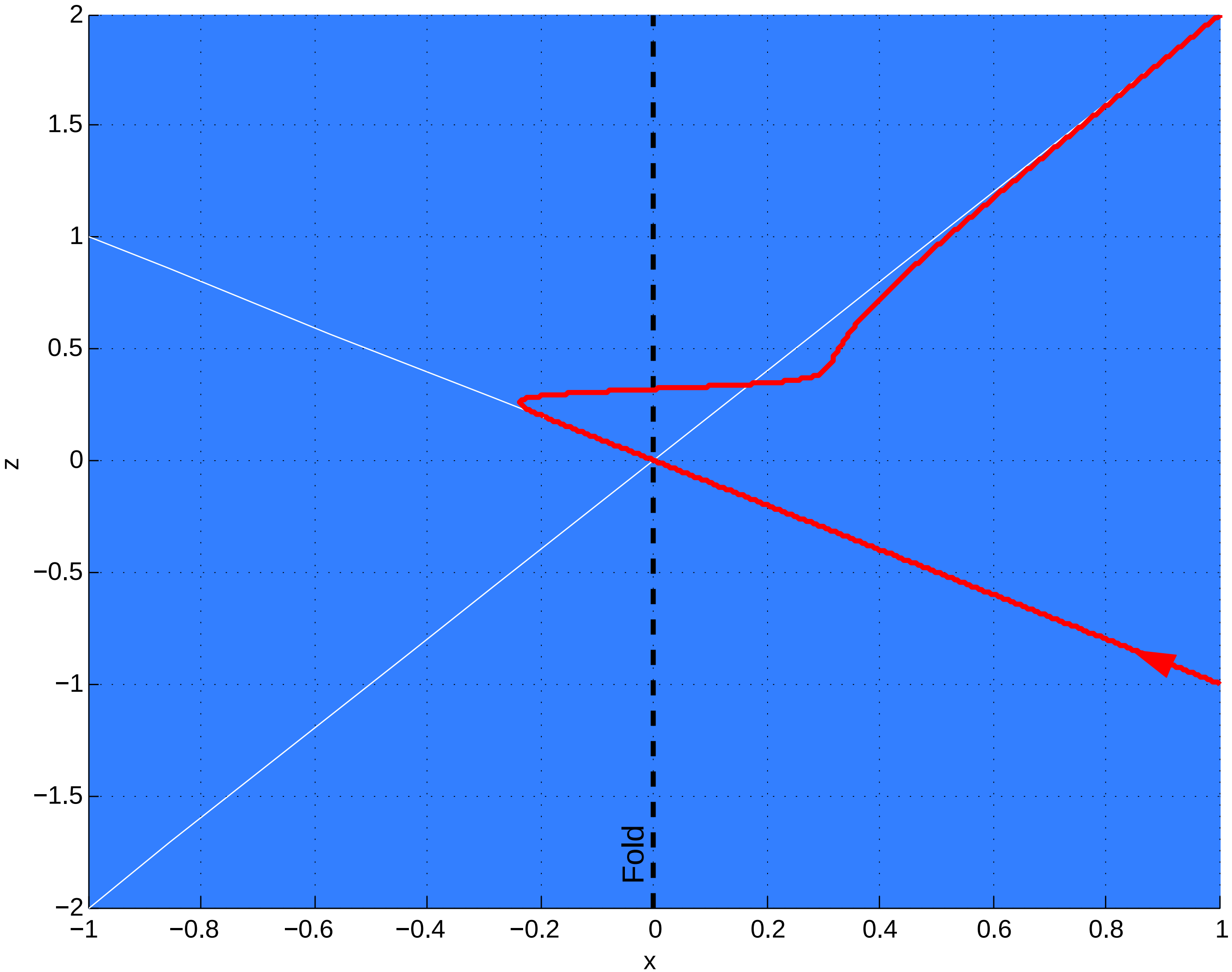

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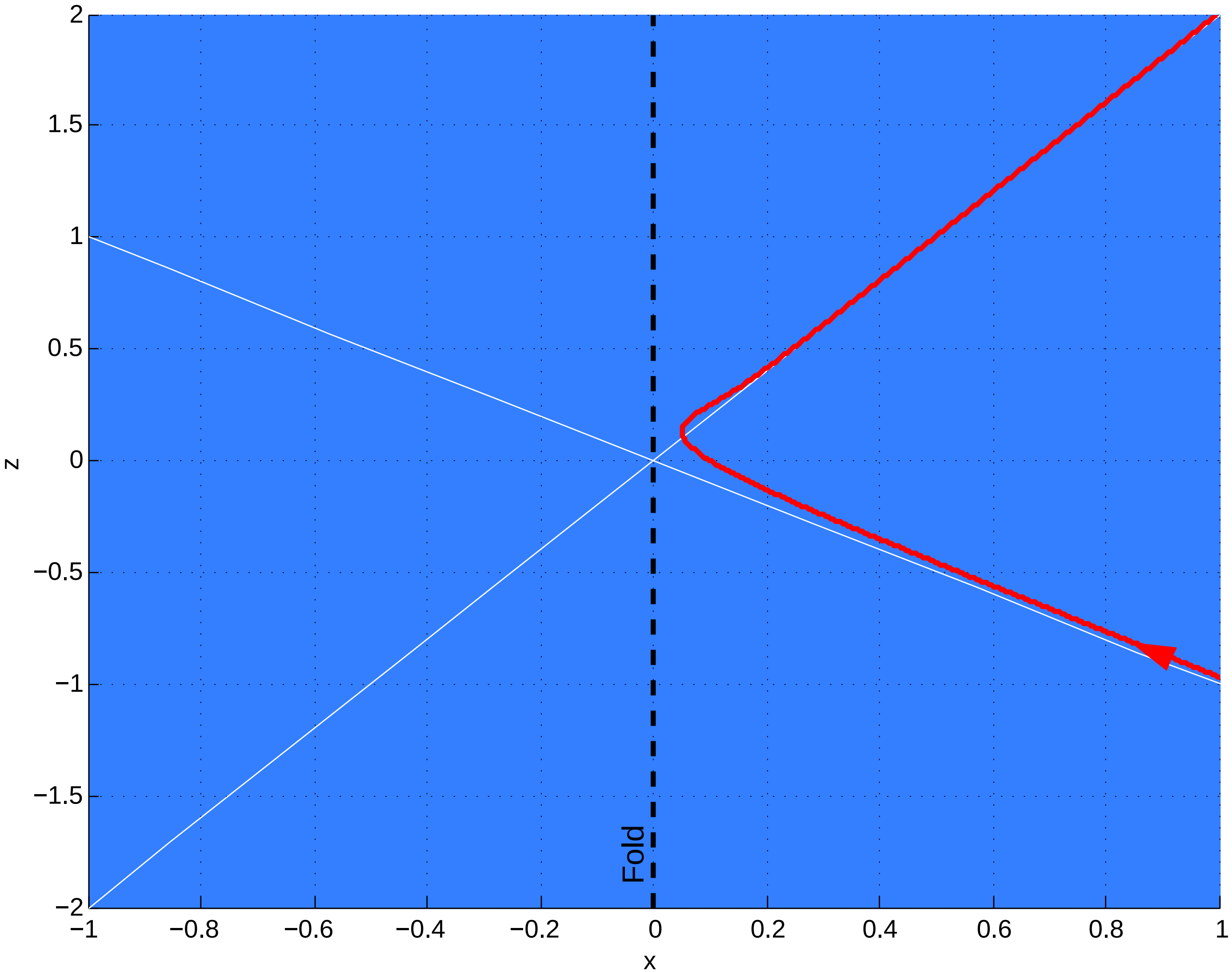

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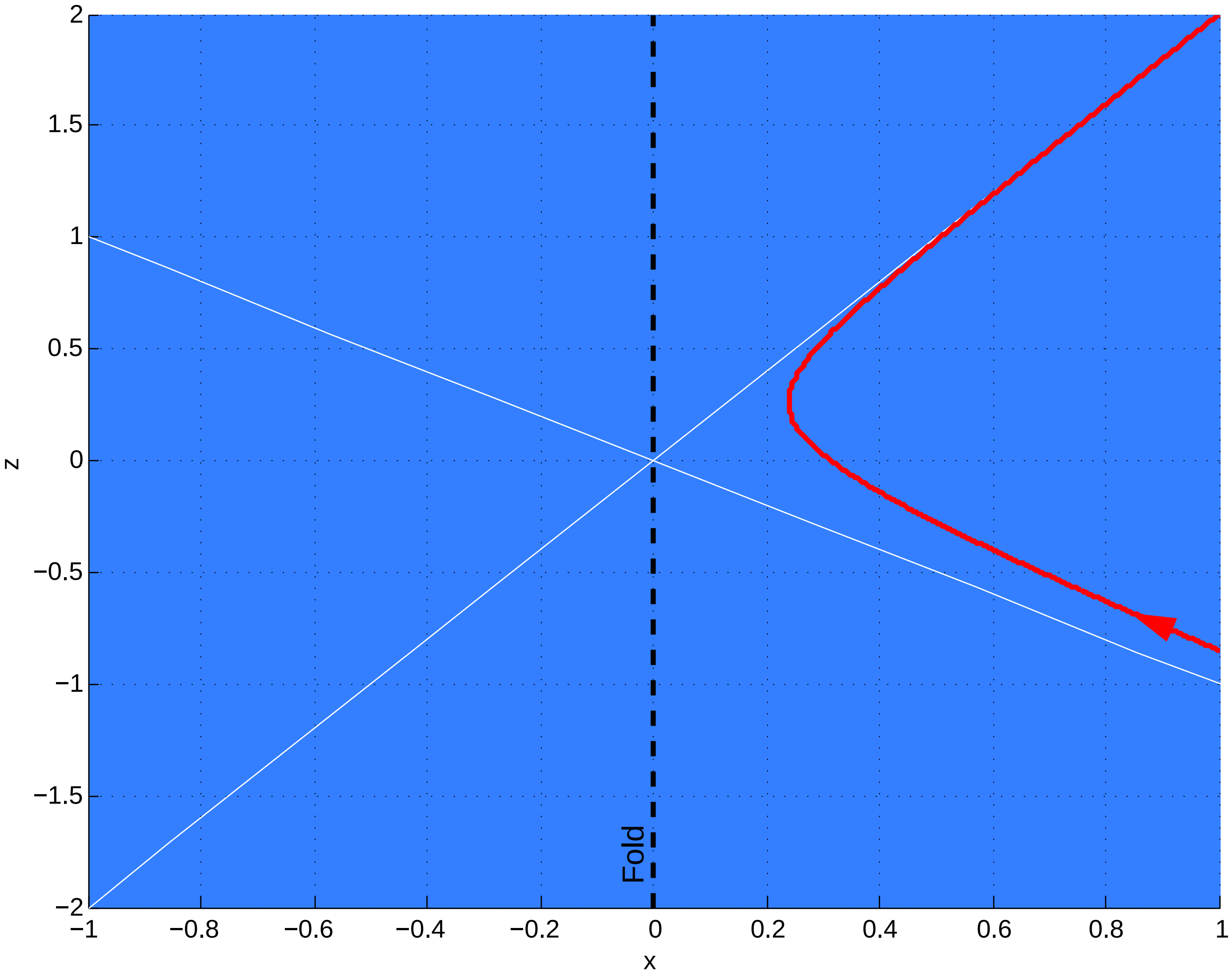

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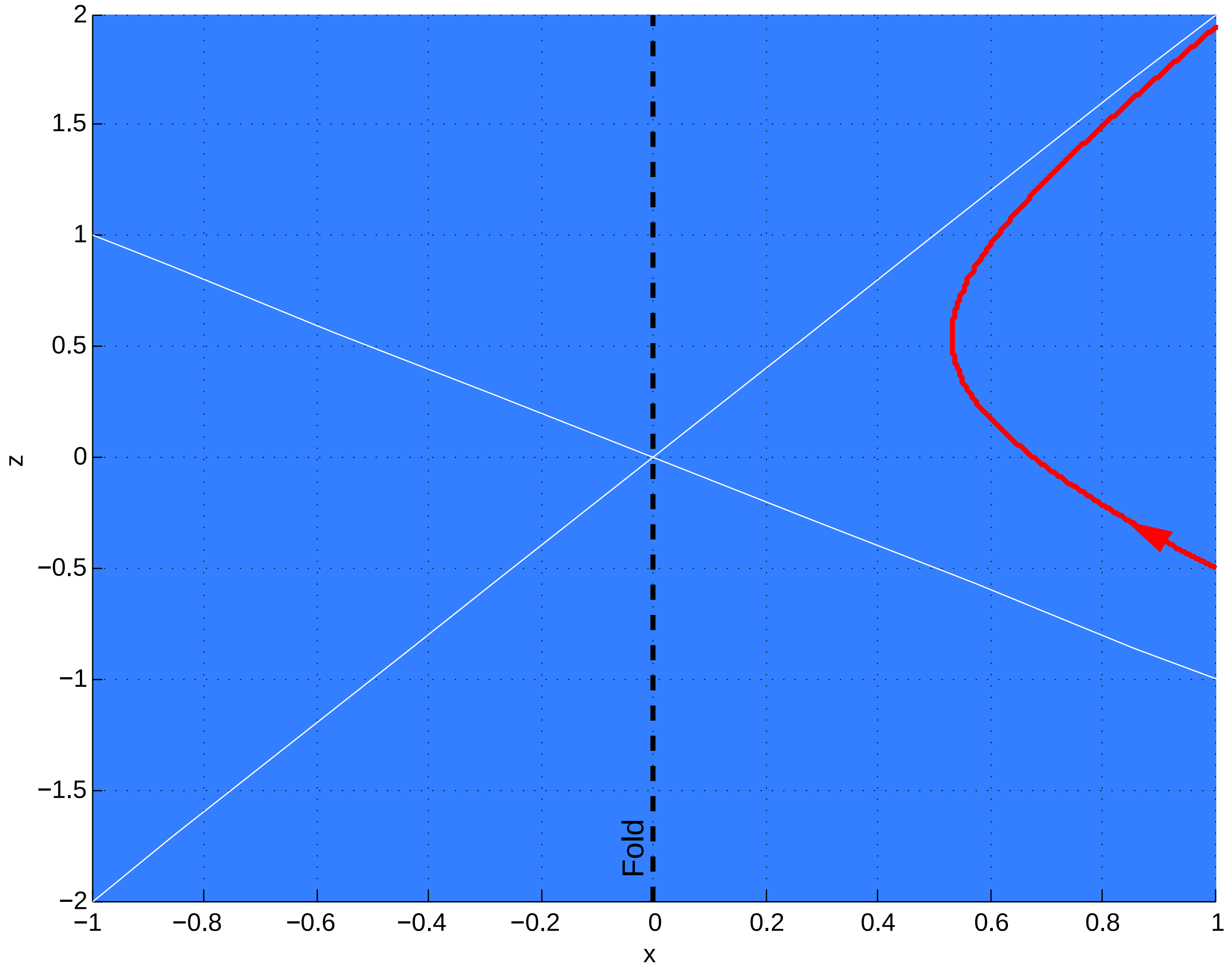
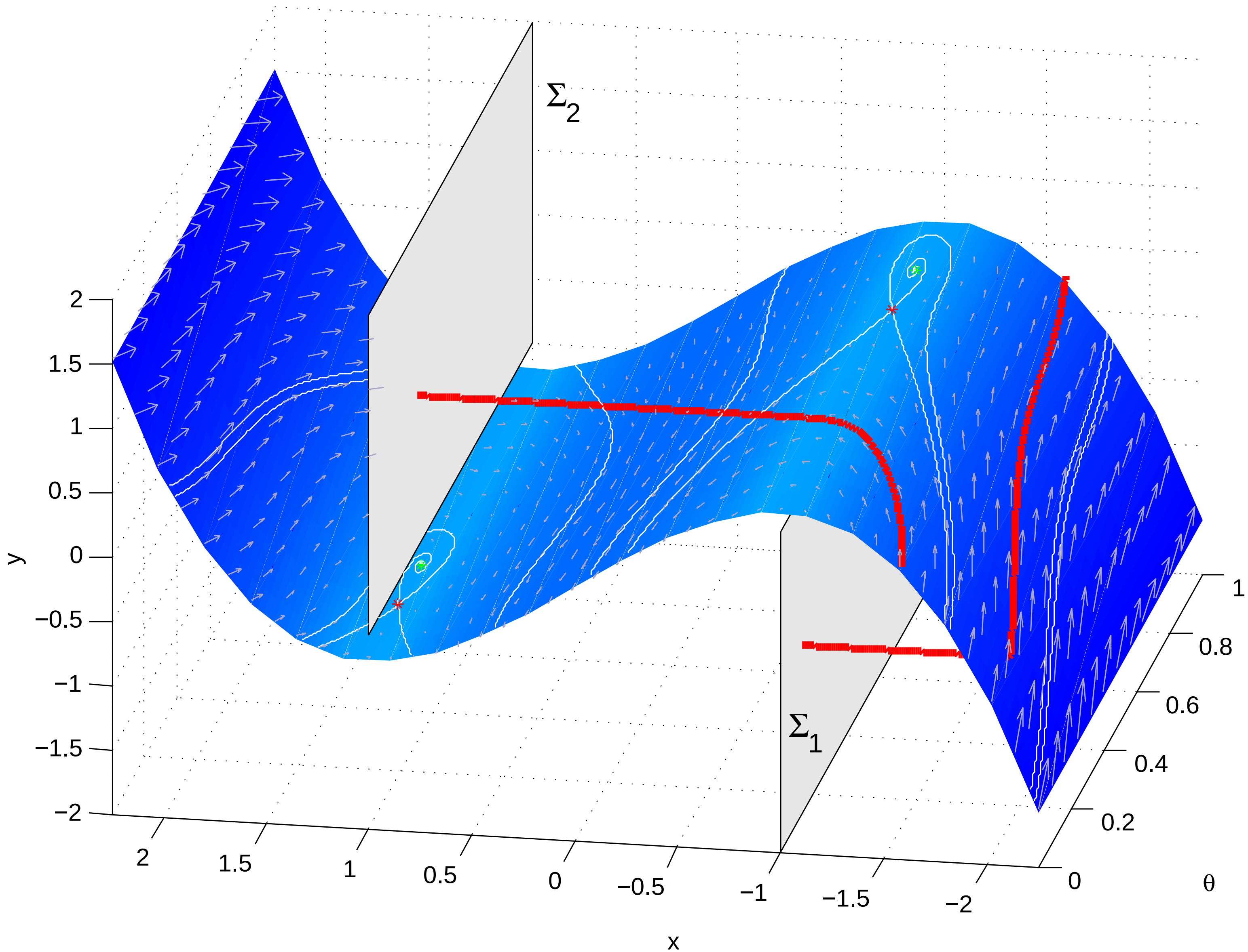


Forced van der Pol Poincaré Map



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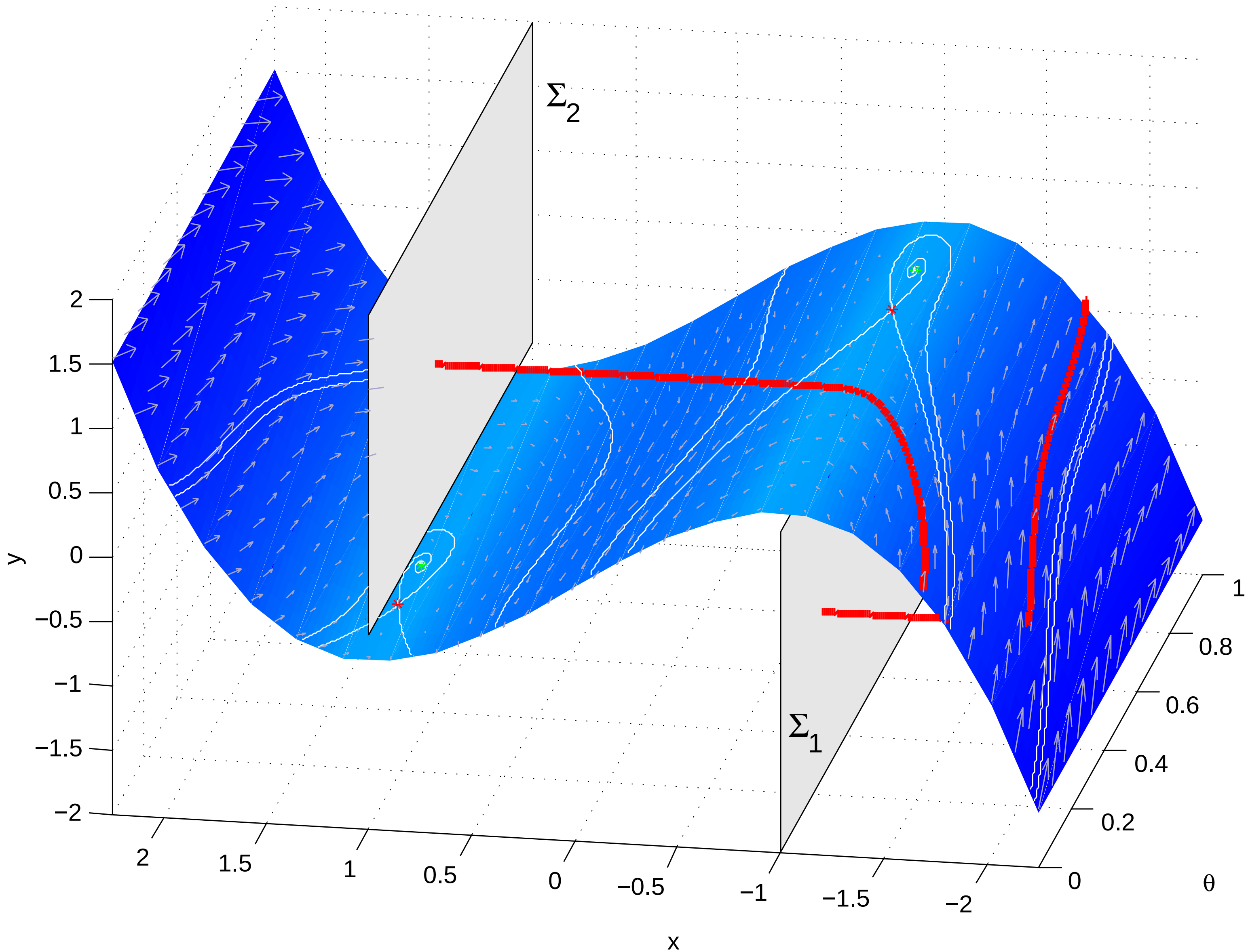

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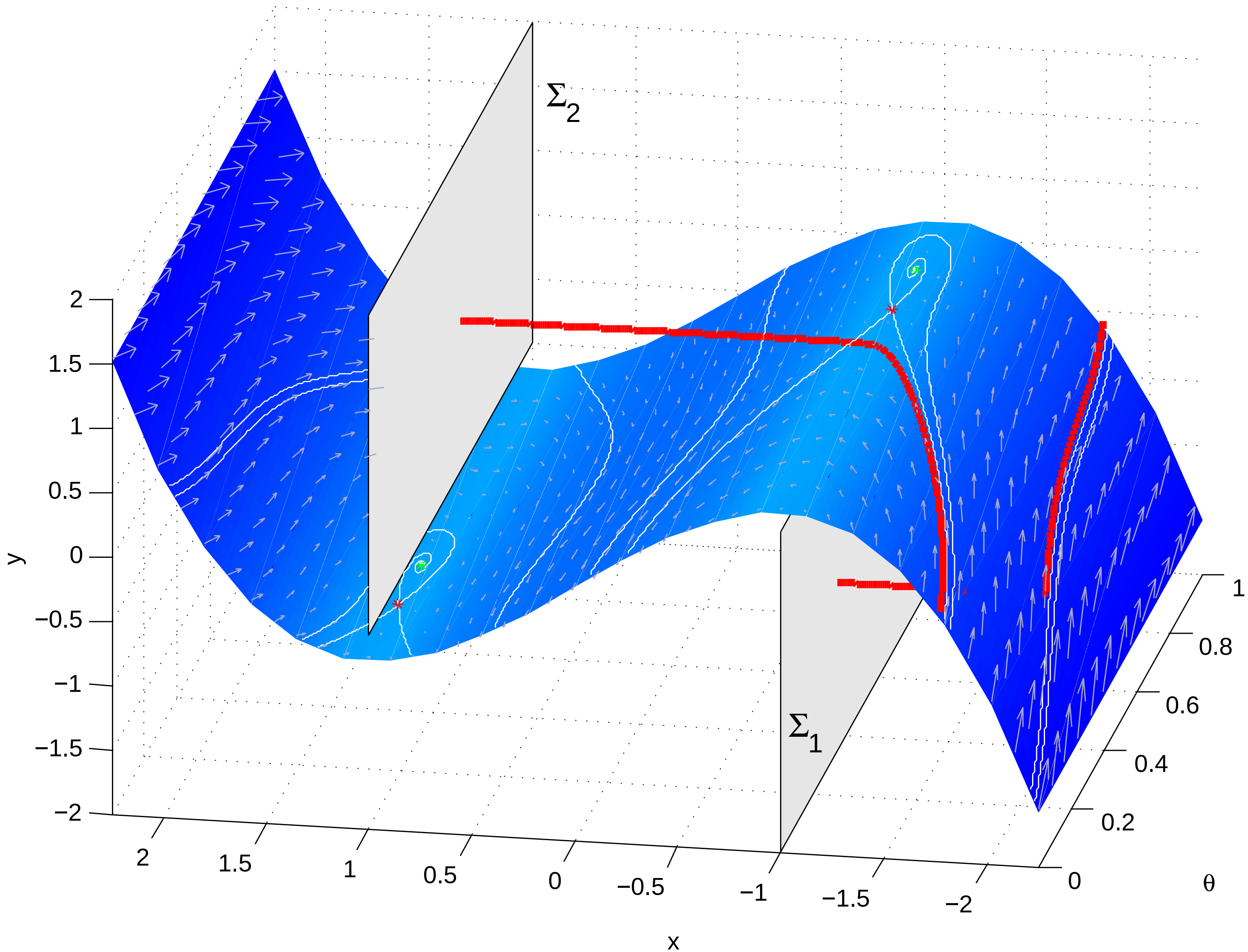

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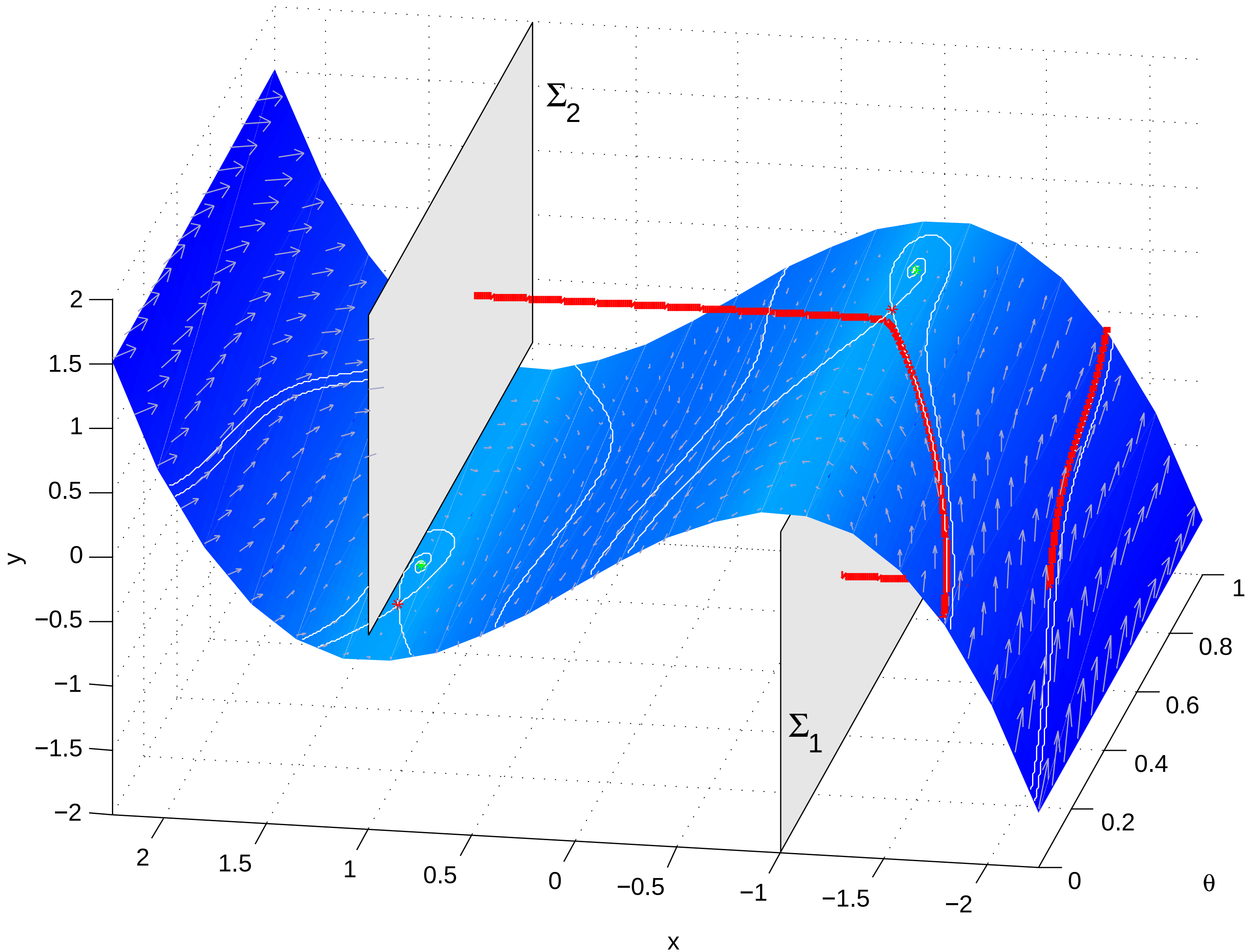

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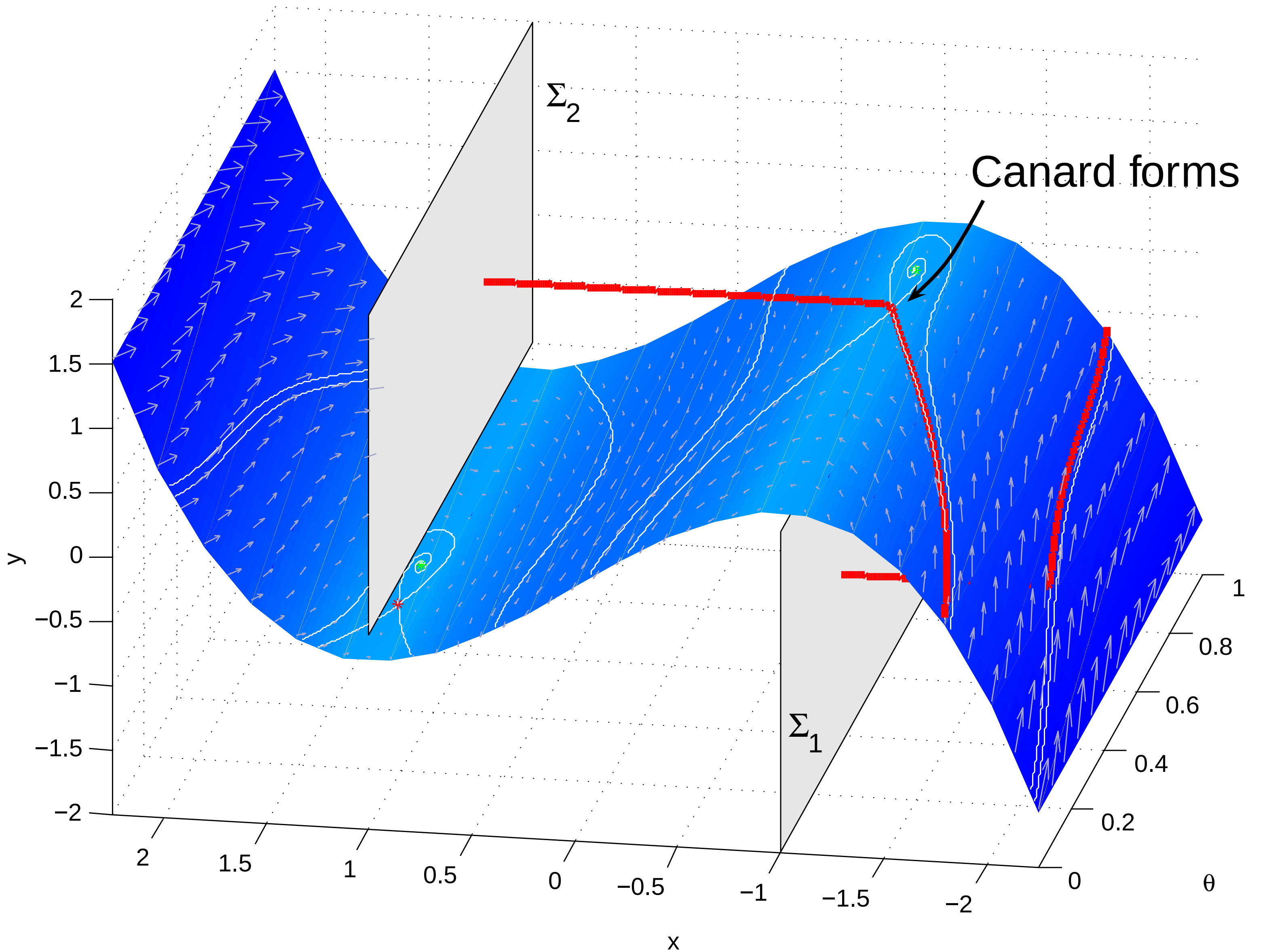

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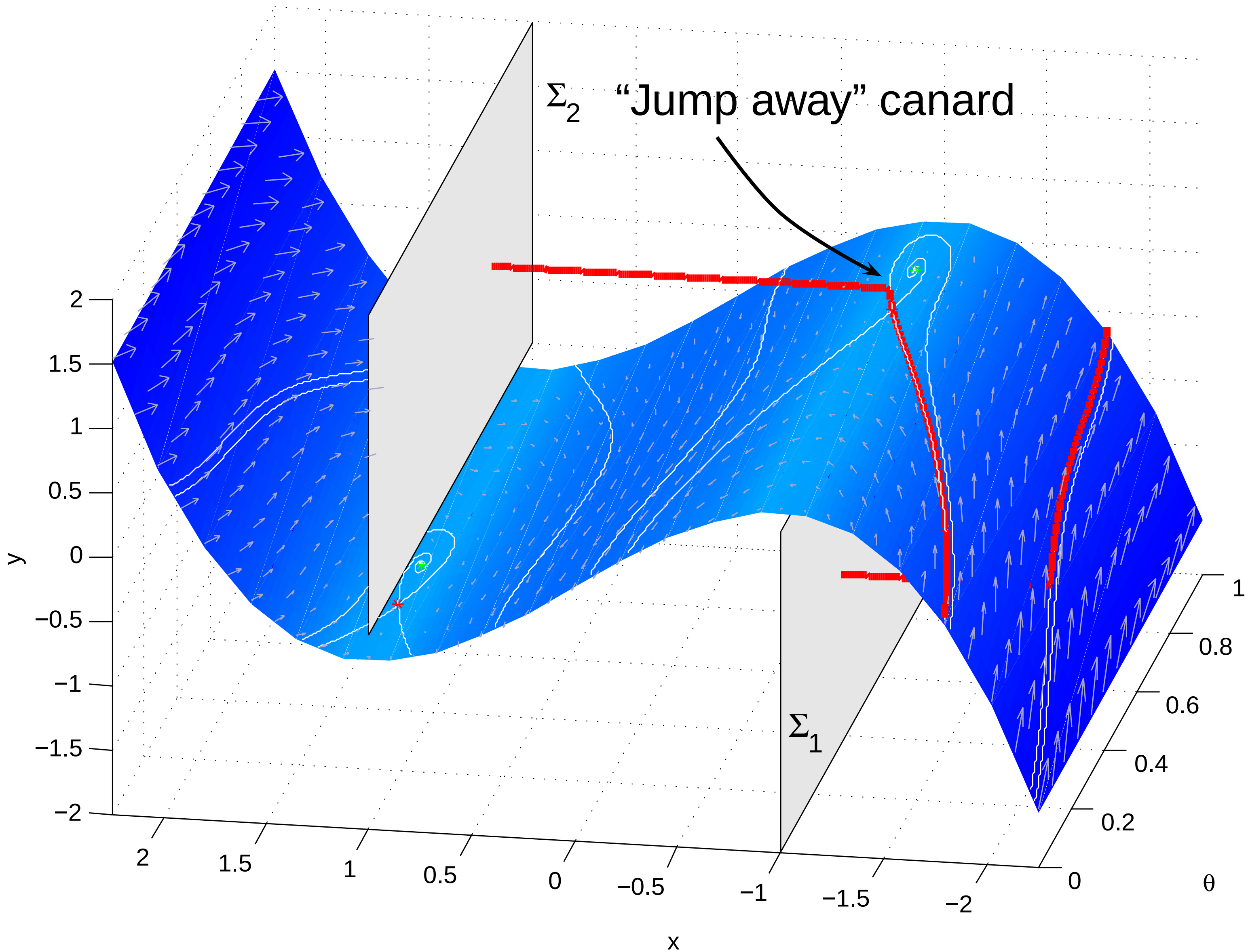

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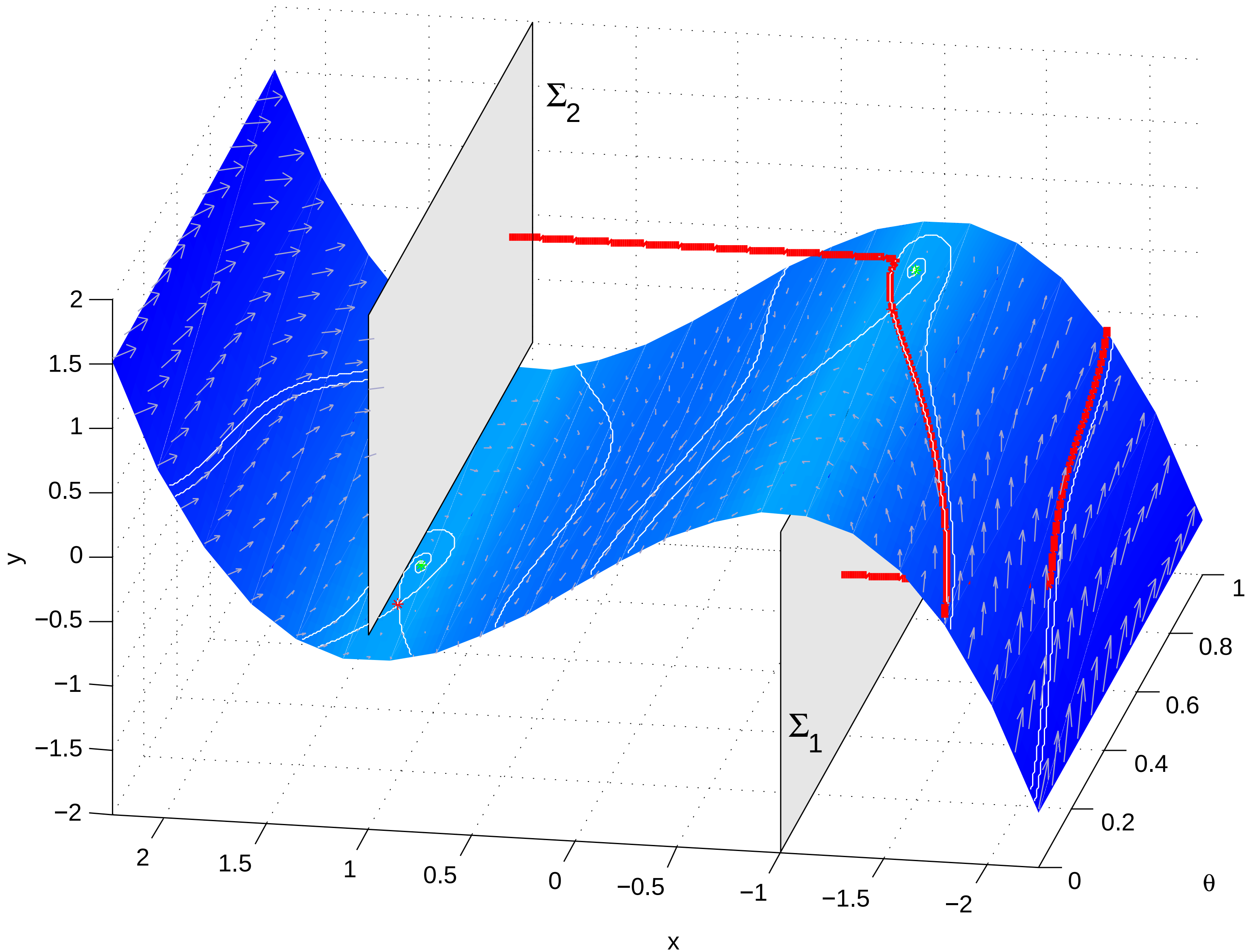

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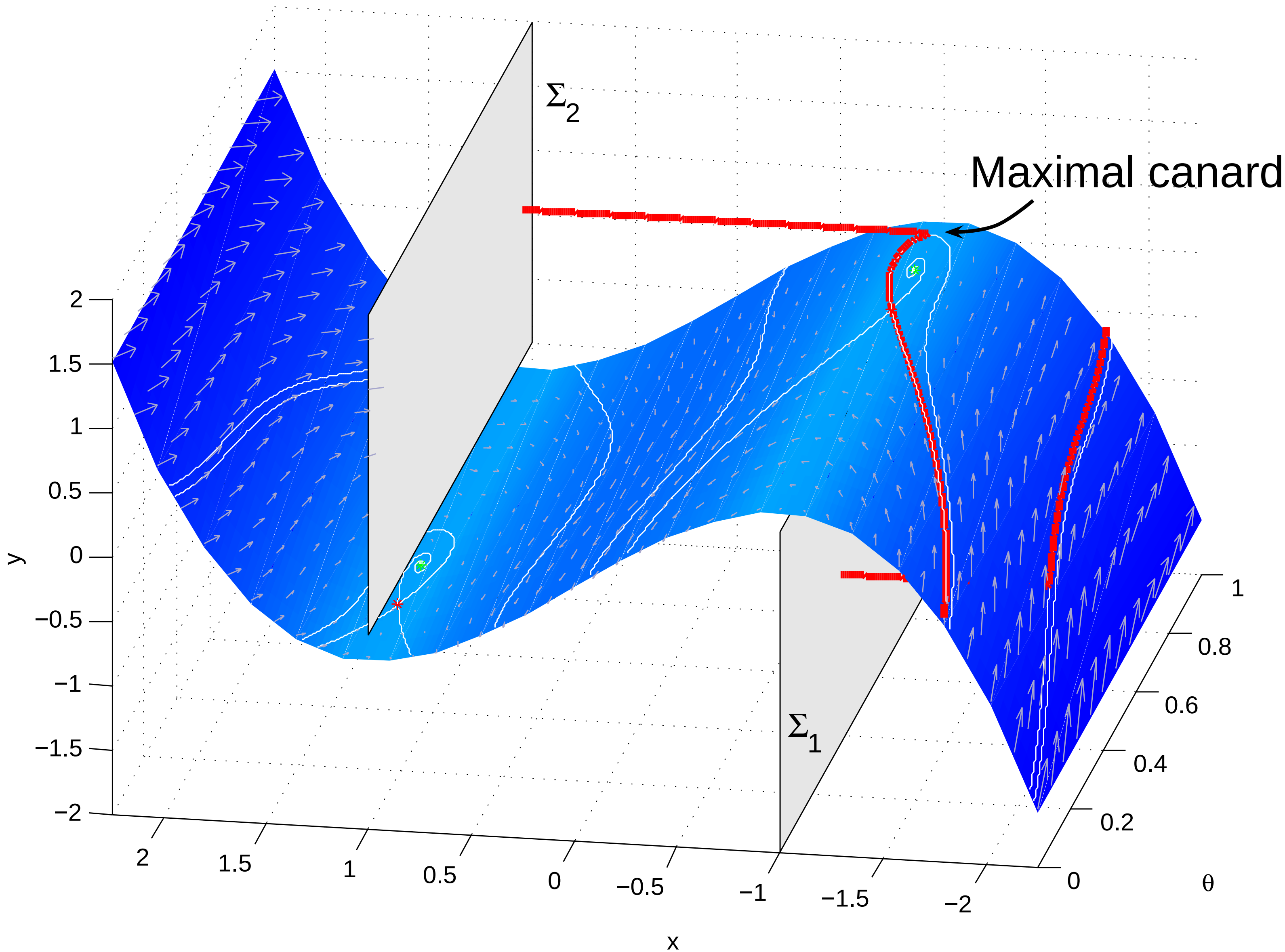

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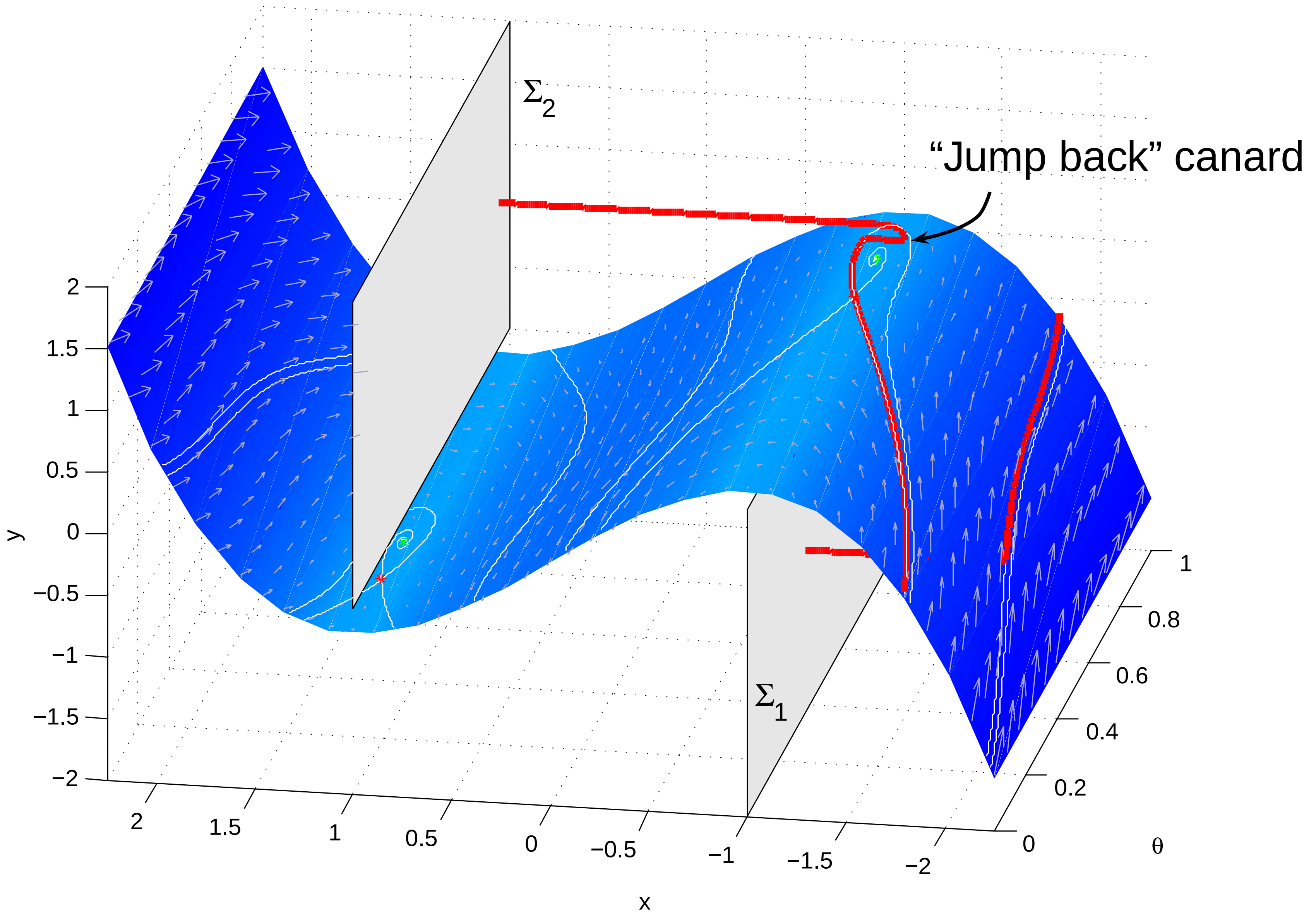

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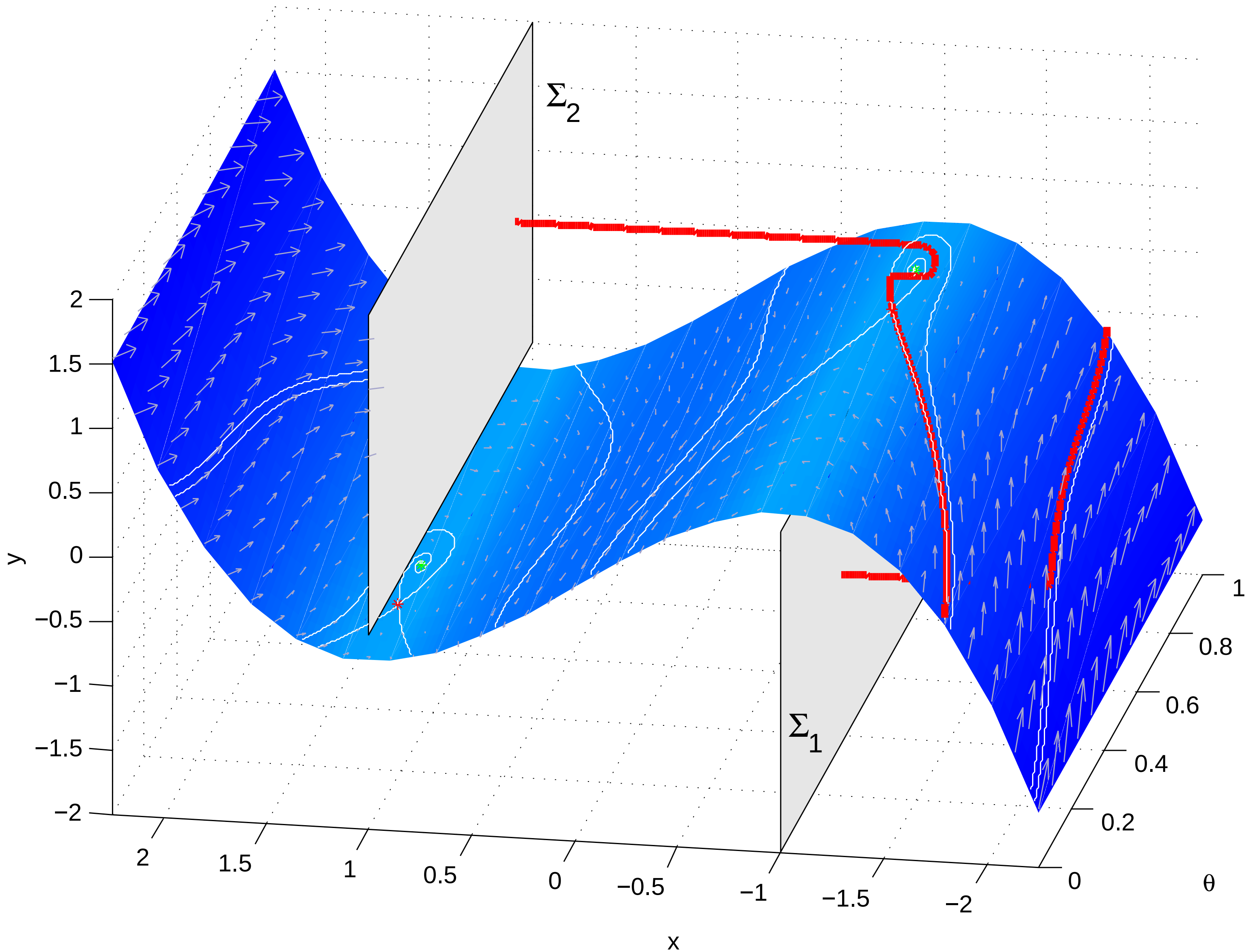

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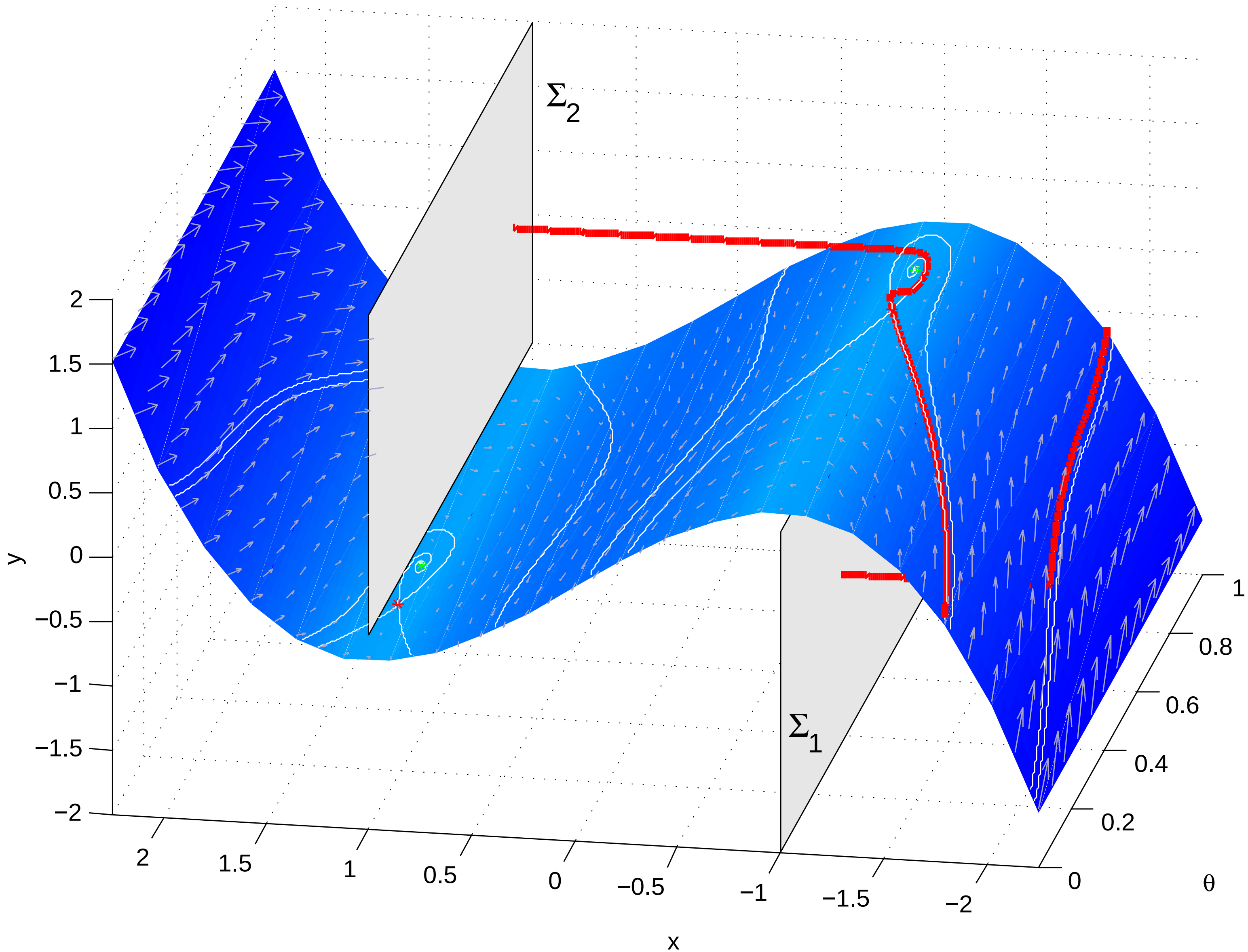

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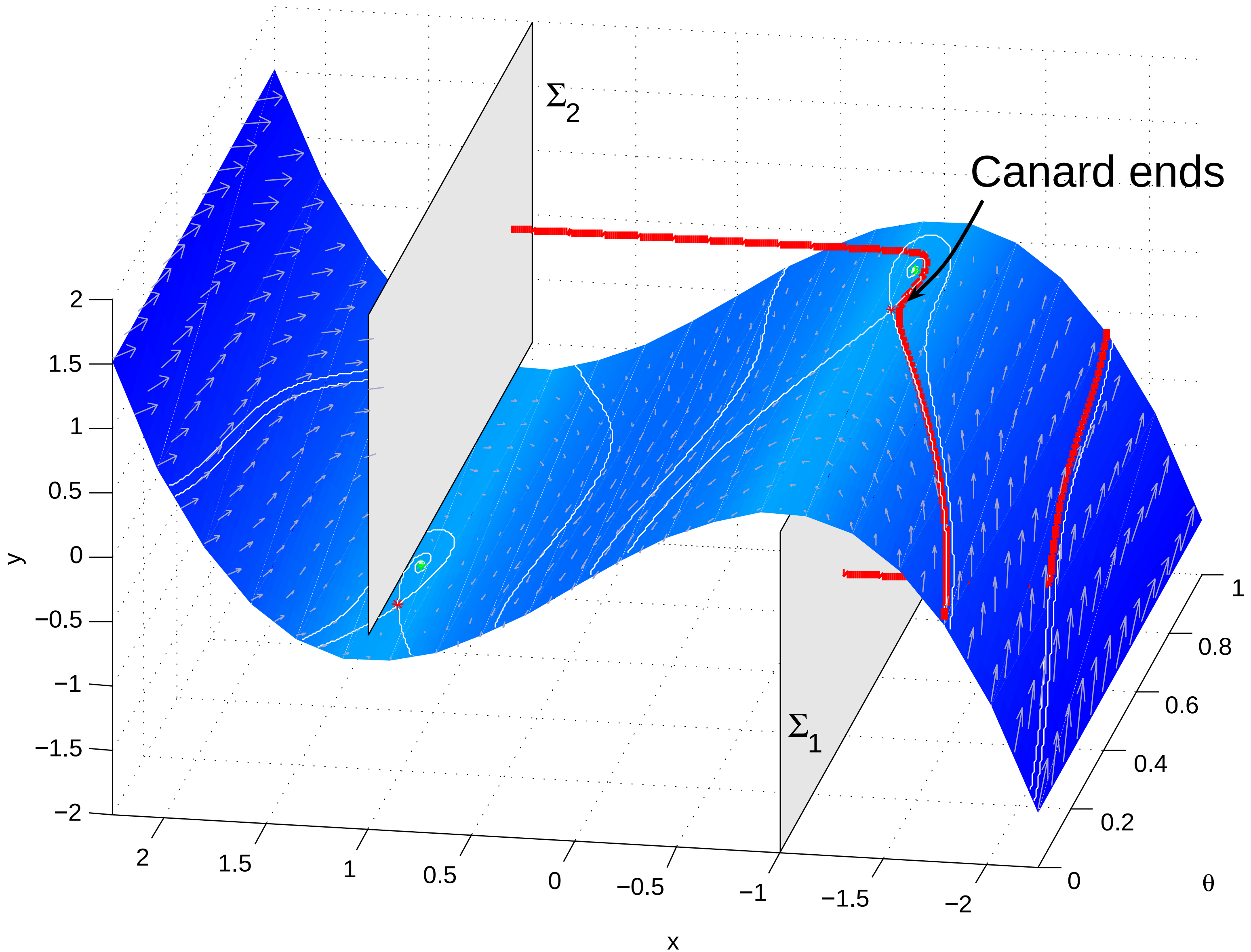

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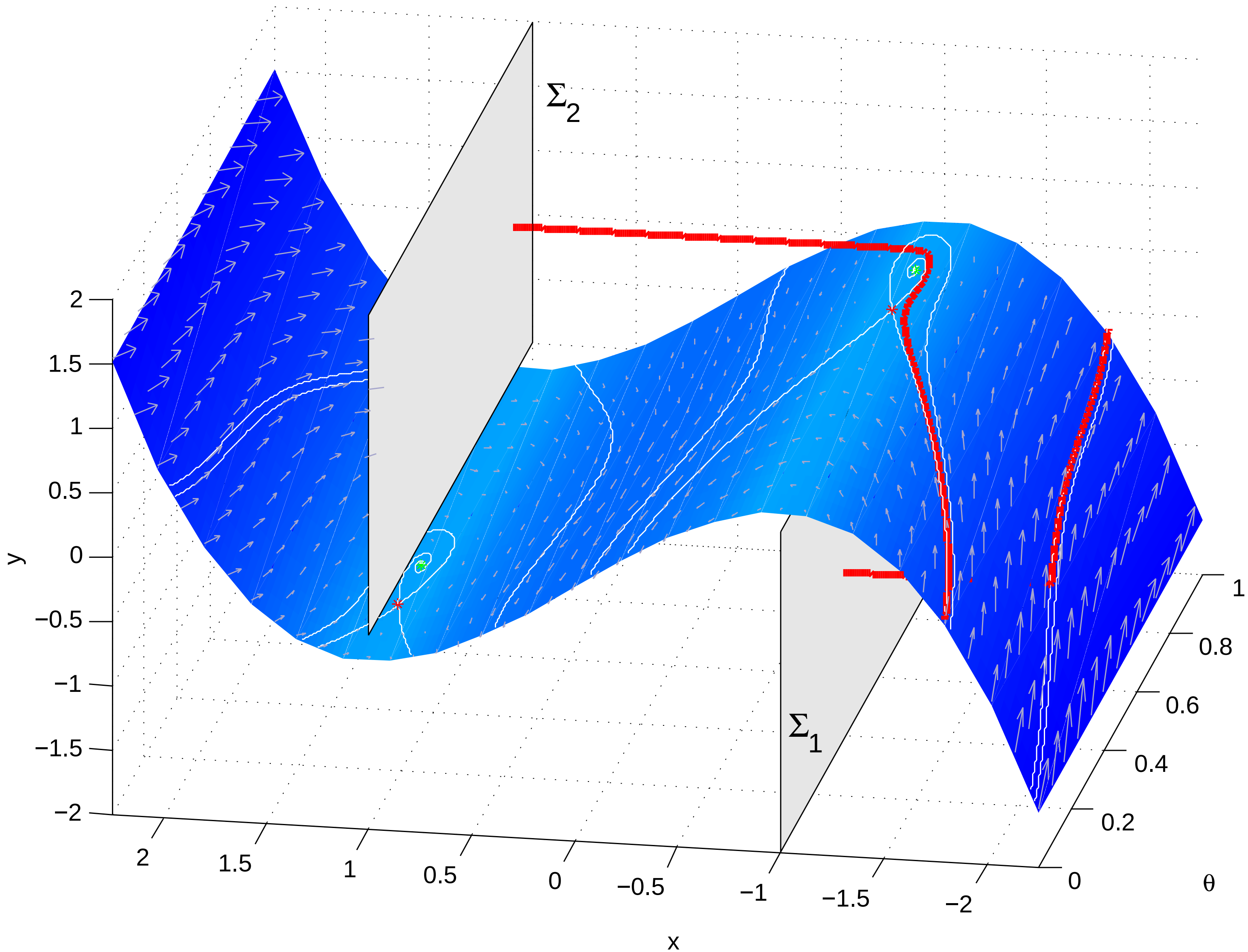

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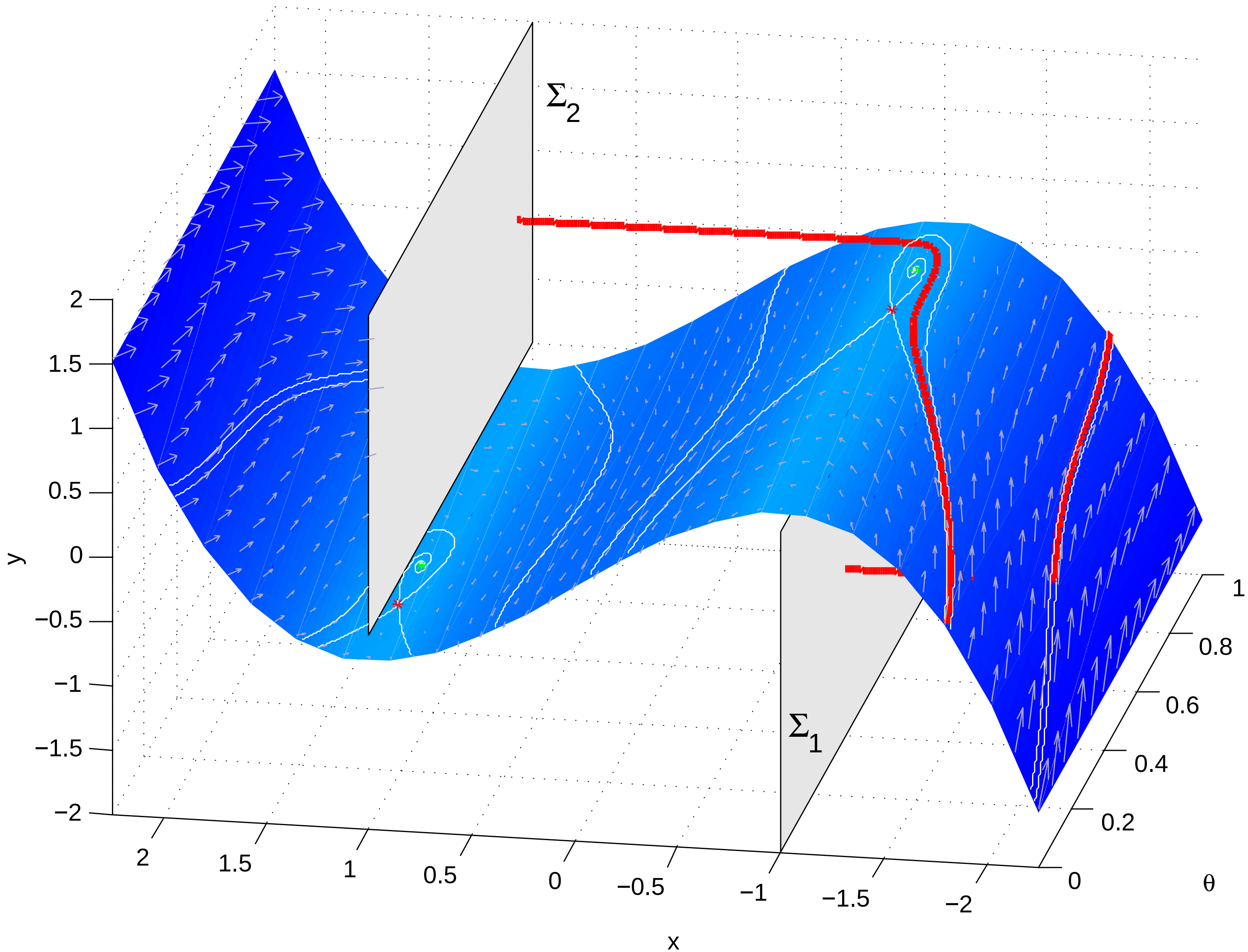

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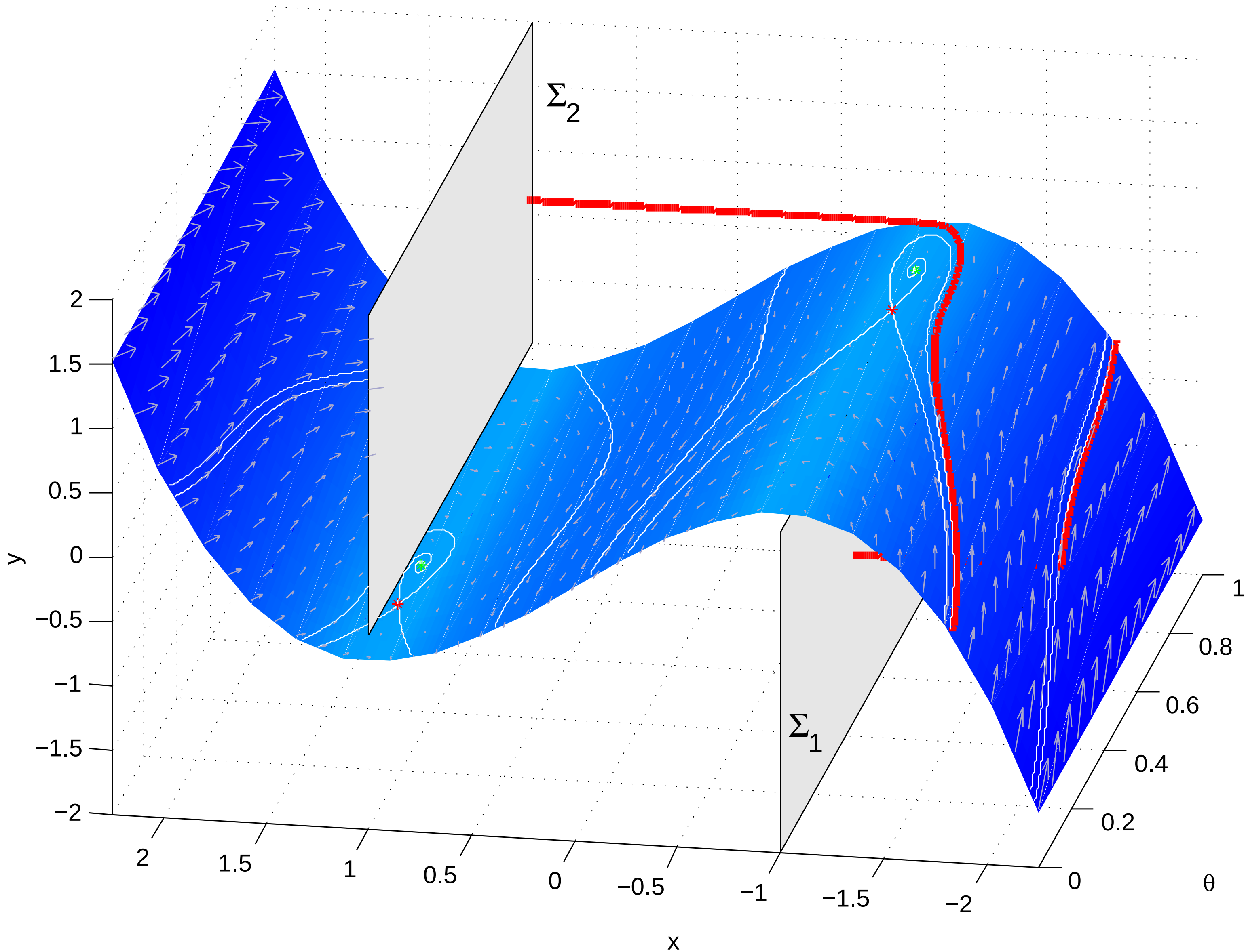

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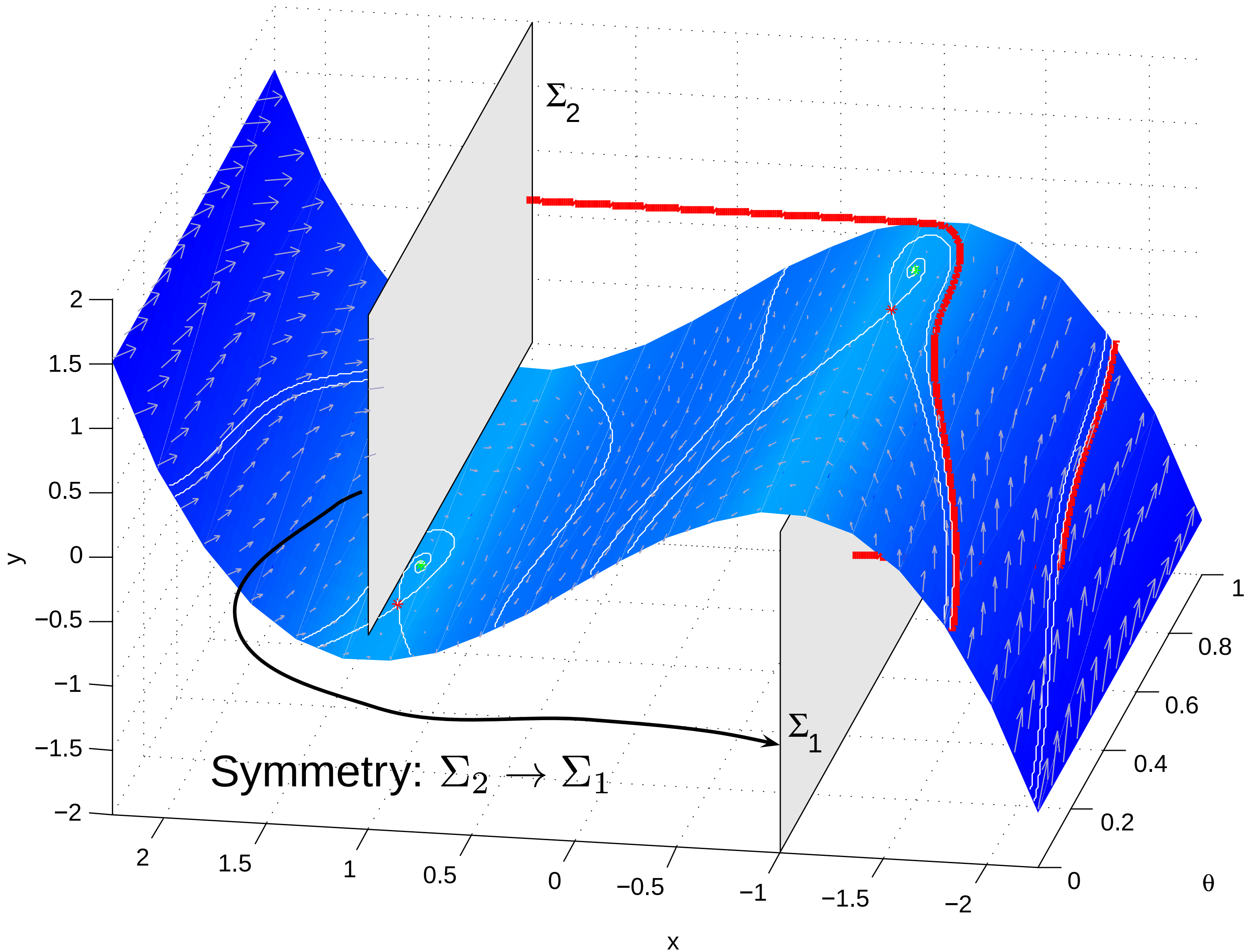
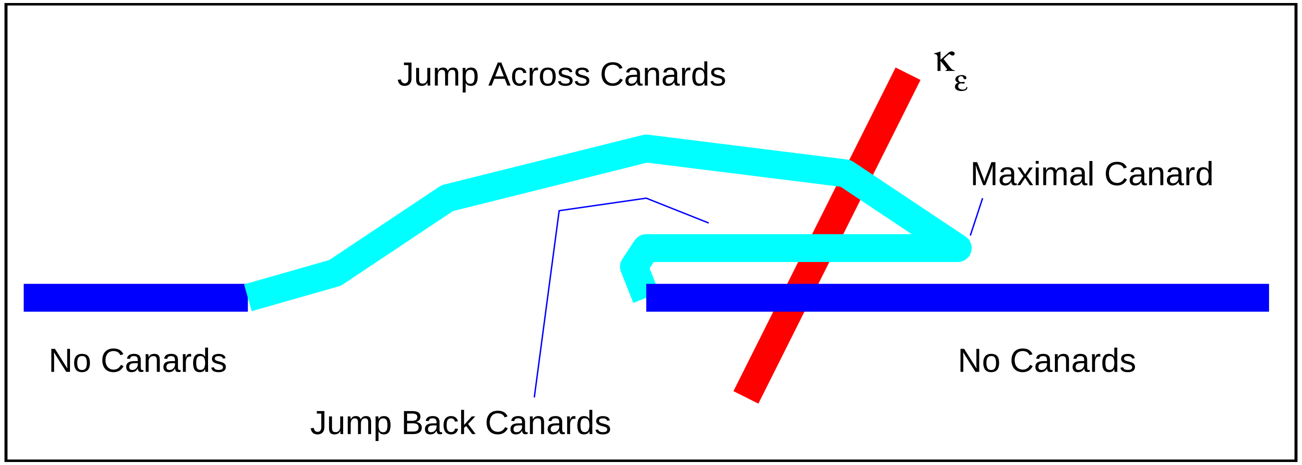


Horseshoe in the Forced van der Pol System

“Cartoon”



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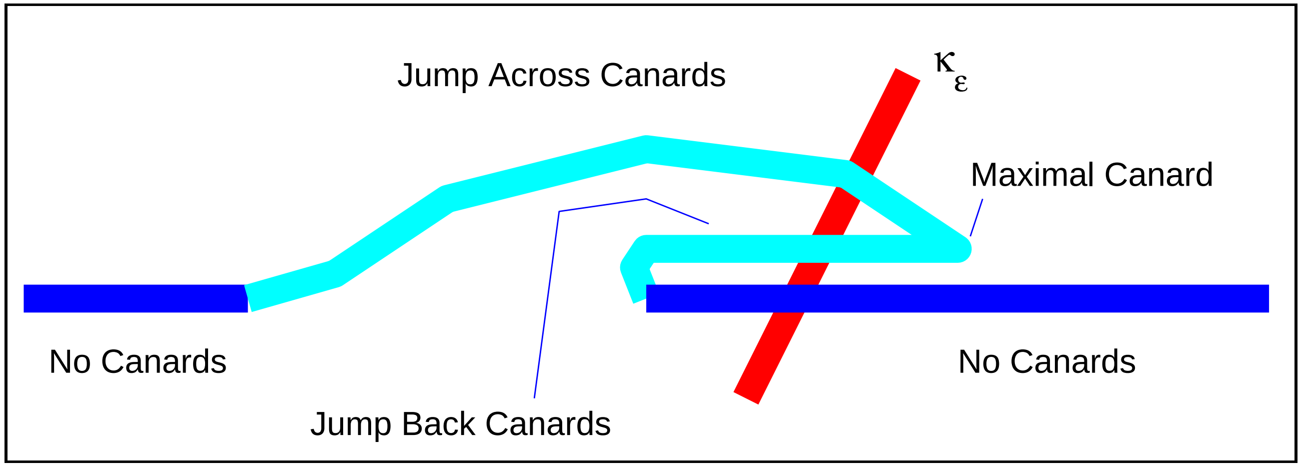
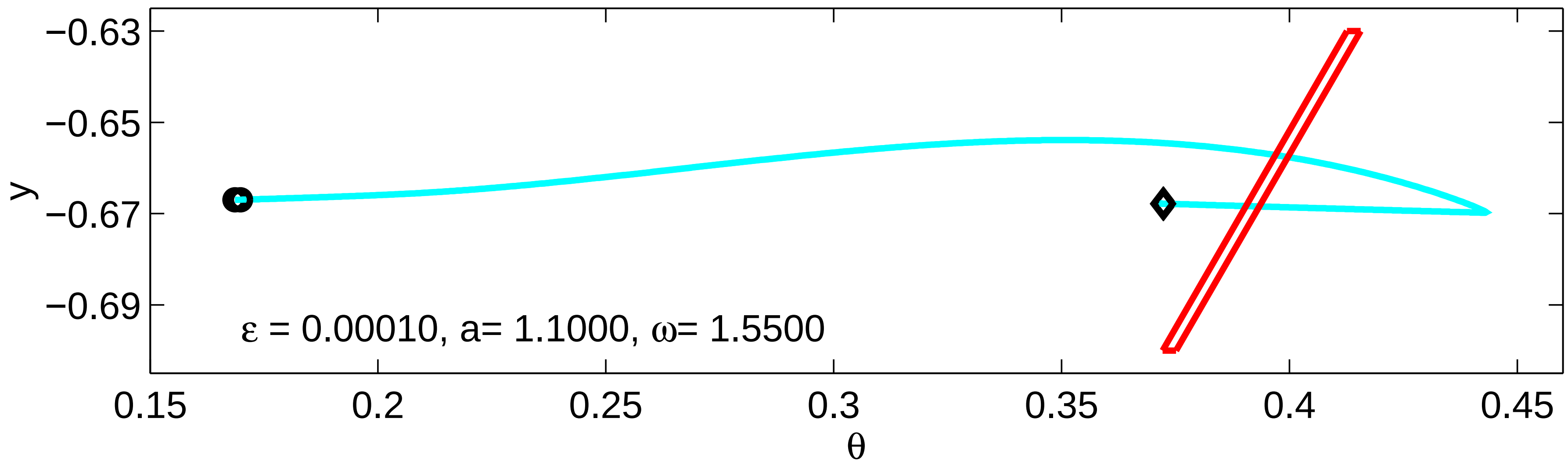
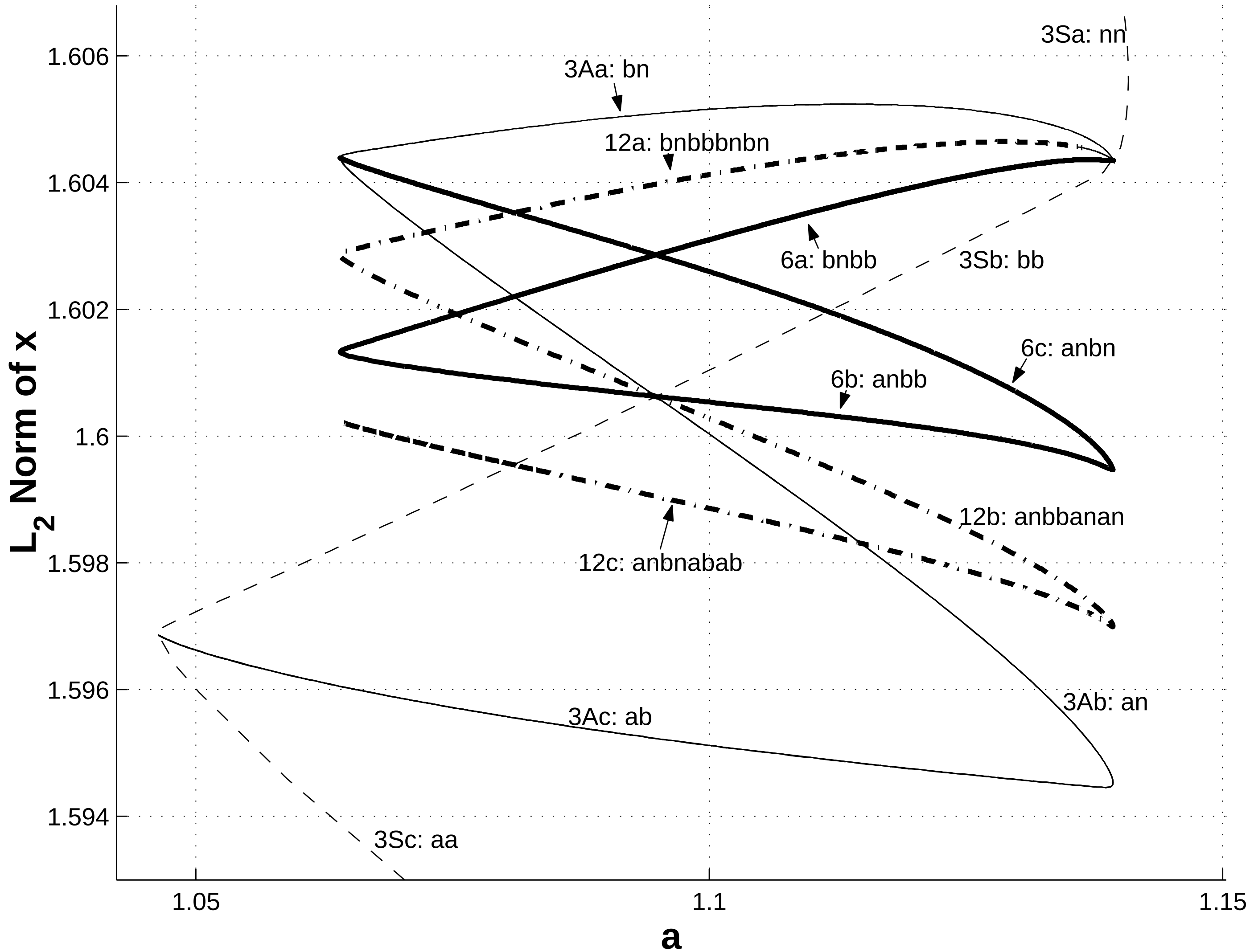


Numerical Computation



Bifurcations of Periodic Orbits



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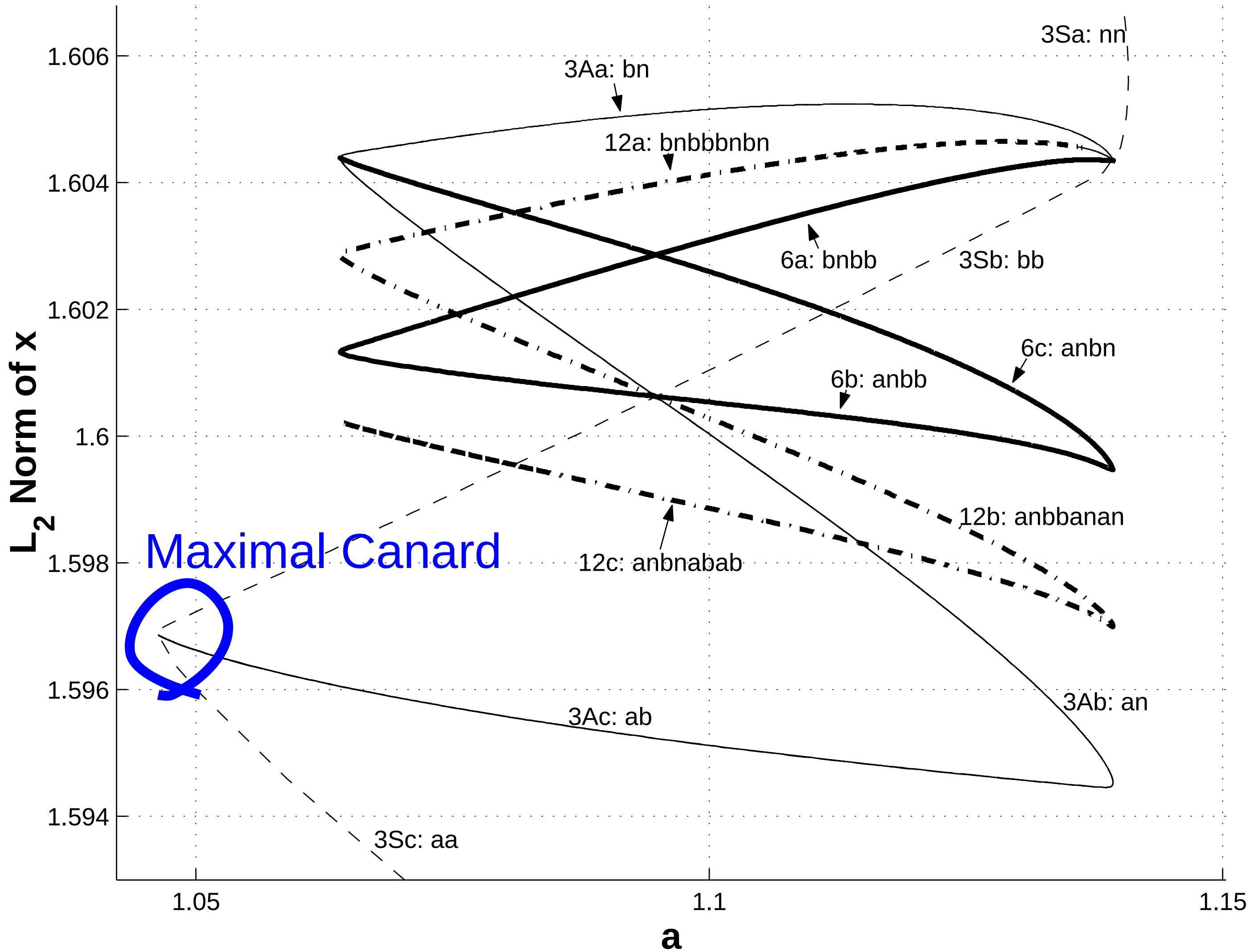

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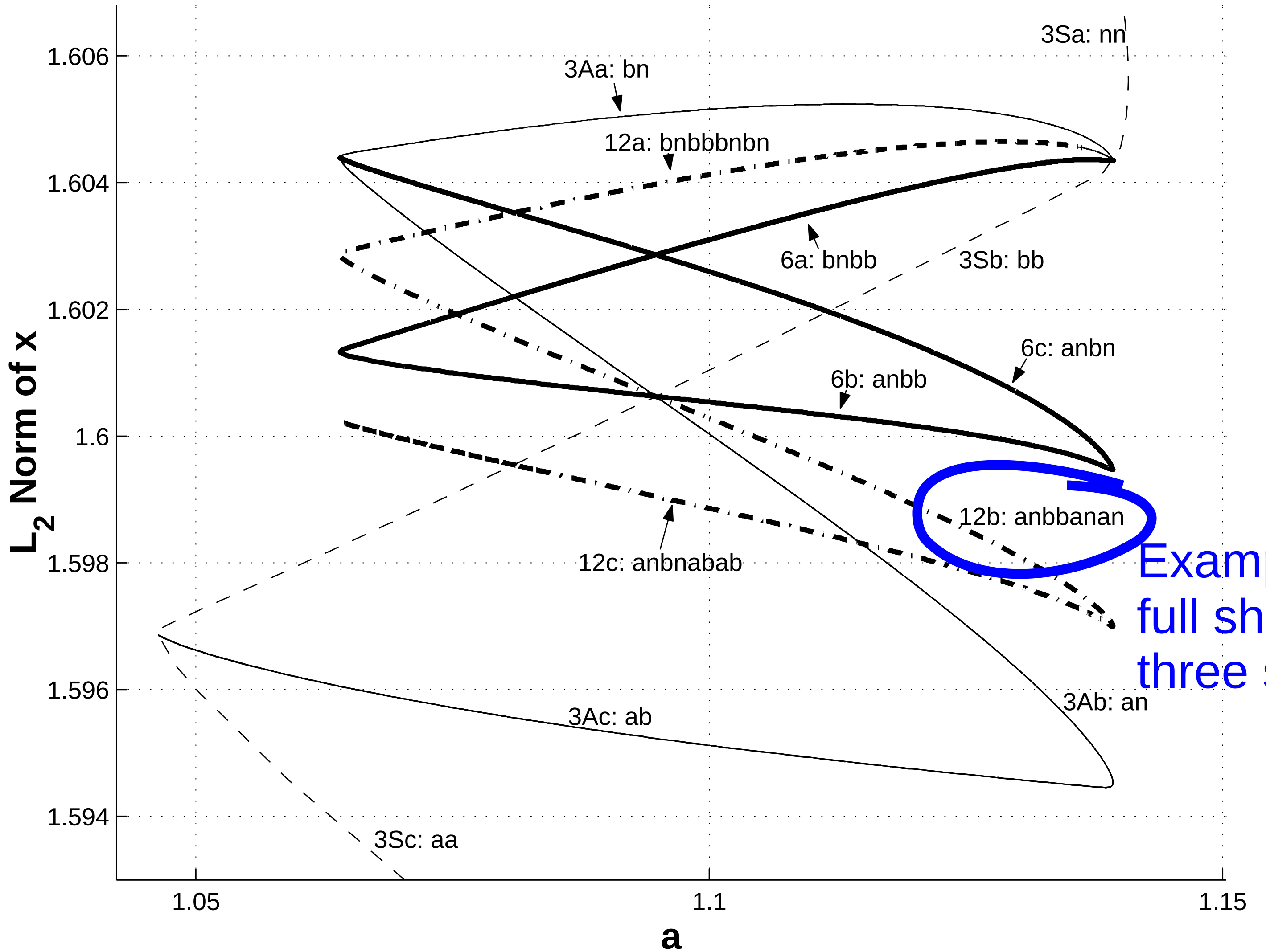
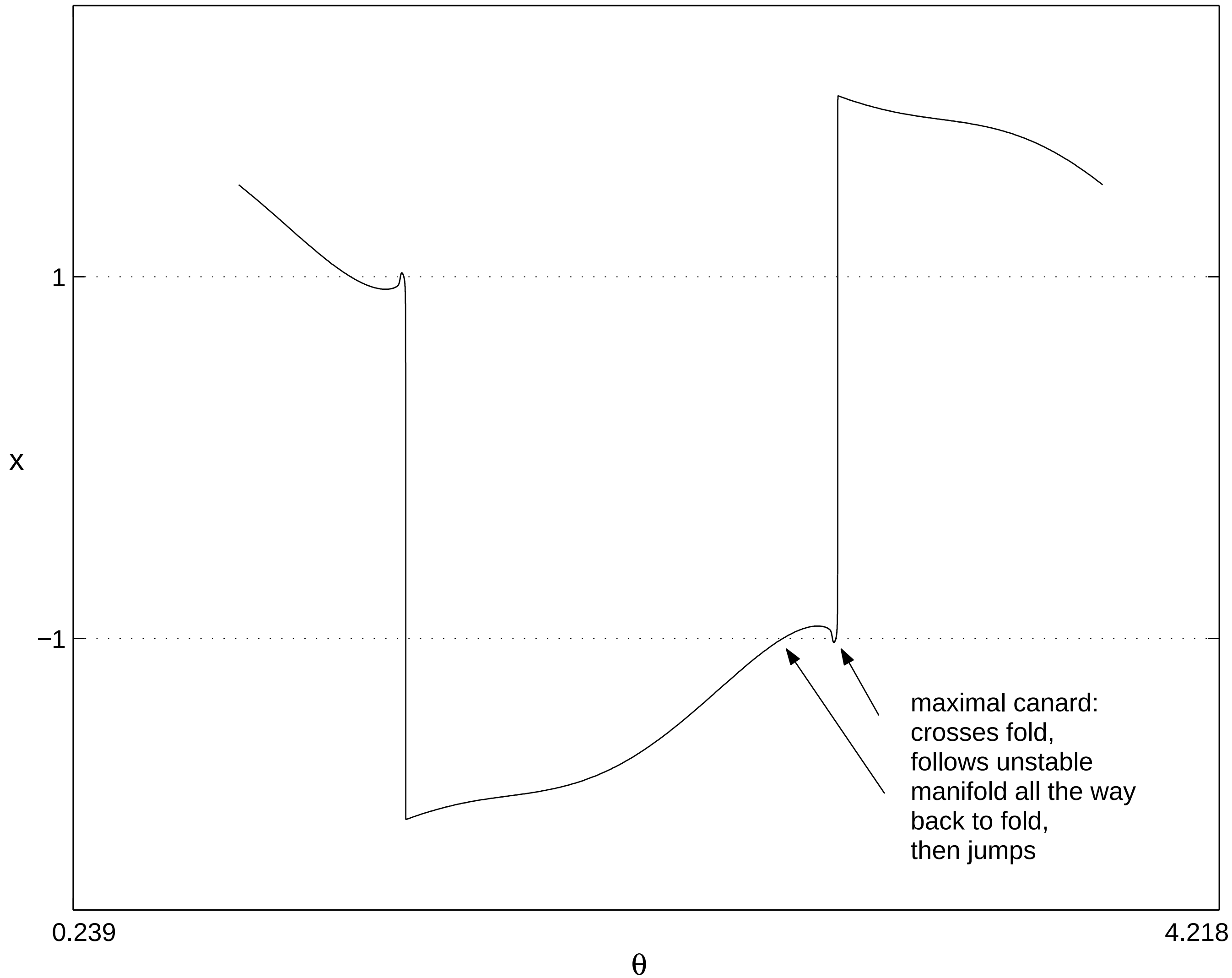


Example from full shift on three symbols

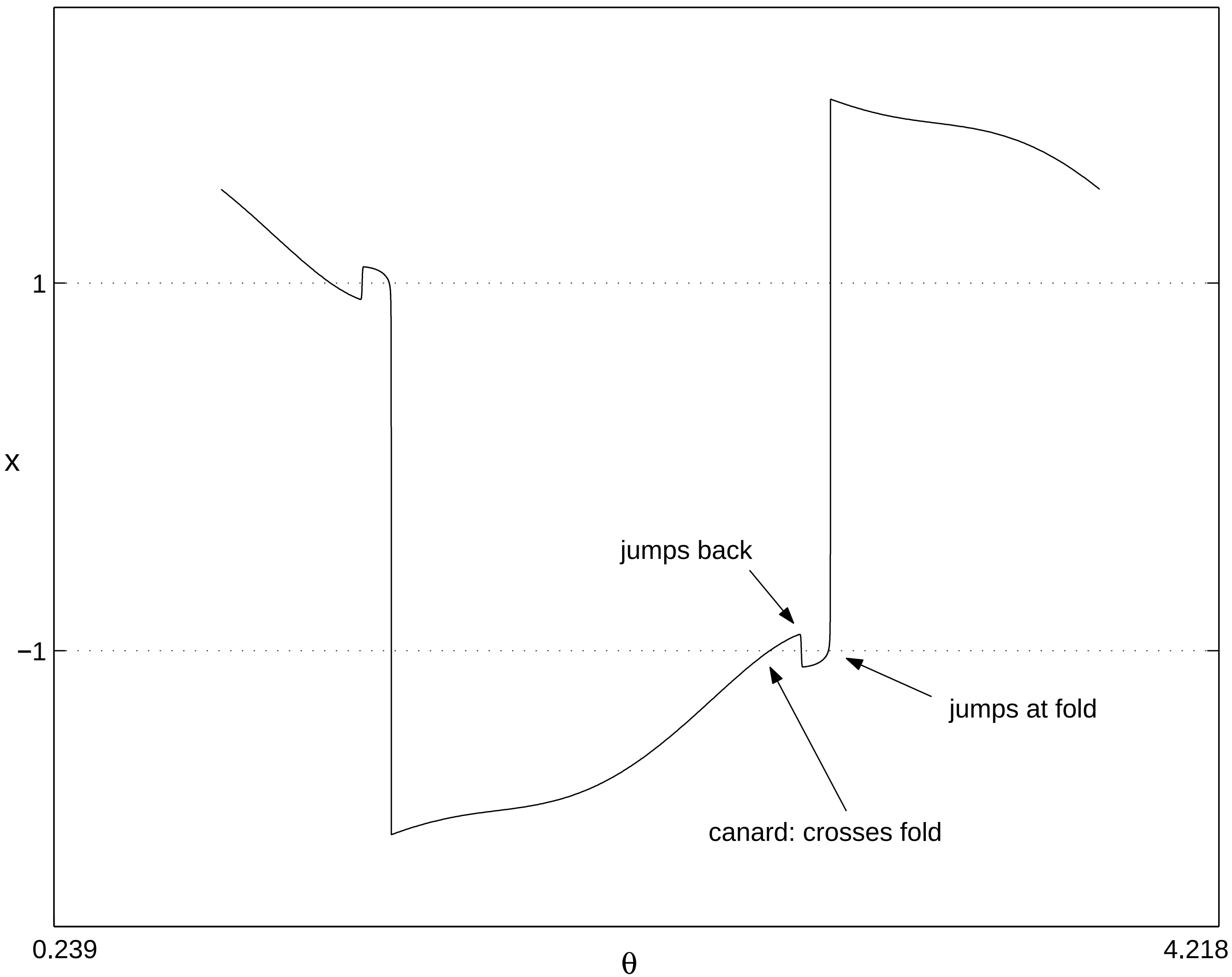
Example: Maximal Canard Bifurcation

Periodic orbit at the limit point has a maximal canard.



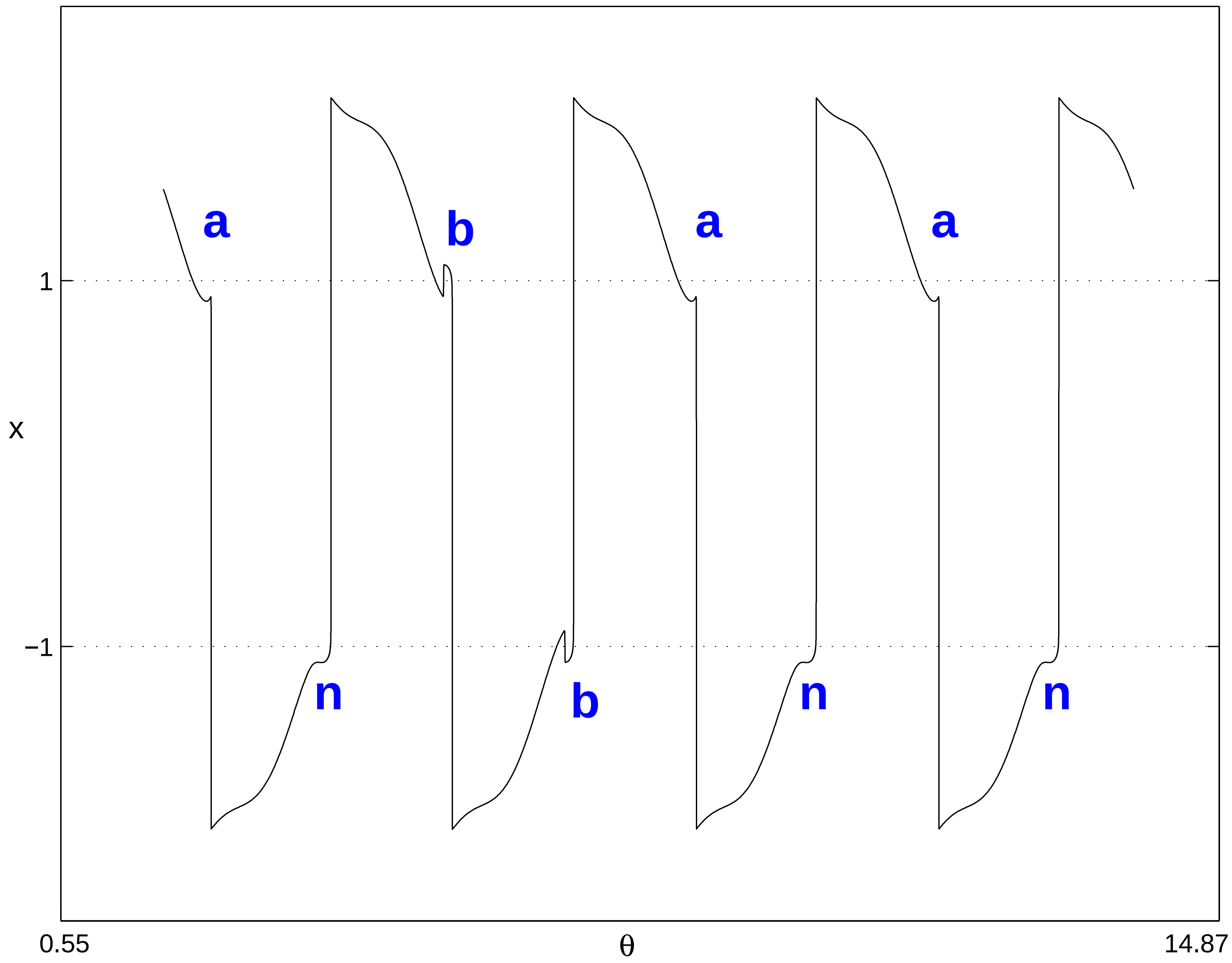
Example: Maximal Canard Bifurcation

Periodic orbit on a branch leading to the left-most limit point (symbols **bb**).



Example: Periodic Orbits and Symbolic Dynamics

Periodic orbit with symbol sequence **anbbanan**.



Summary

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- A detailed numerical study combined with an understanding of the fast/slow dynamics provides a clear picture of the horseshoe map in the forced van der Pol system.

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- A detailed numerical study combined with an understanding of the fast/slow dynamics provides a clear picture of the horseshoe map in the forced van der Pol system.
- *Canards* that form at nonhyperbolic points of the slow manifold play a key role in this picture.

References

- ☞ K. Bold, C. Edwards, J. Guckenheimer, S. Guharay, K. Hoffman, J. Hubbard, R. Oliva, and W. Weckesser, **The forced van der Pol equation II: canards in the reduced system**, to appear in *SIAM J. Appl. Dyn. Sys.*
- ☞ J. Guckenheimer, K. A. Hoffman, and W. Weckesser, **The forced van der Pol equation I: the slow flow and its bifurcations**, *SIAM J. Appl. Dyn. Sys.* **2**(1) 1-35 (2003).
- ☞ J. Guckenheimer, K. Hoffman, and W. Weckesser, **Global bifurcations of periodic orbits in the forced van der Pol equation**, in *Global Analysis of Dynamical Systems*, H. W. Broer, B. Krauskopf, G. Vegter, eds., Inst. of Physics Publ., Bristol, (2001).
- ☞ J. Guckenheimer, K. Hoffman, and W. Weckesser, **Numerical computation of canards**, *Int. J. Bif. Chaos*, **10**, 2669-2687 (2000)