Reduced System Computing and MMOs

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Acknowledgments

❏ NSF Grant DMS-0514468: *RUI: Reduced System Computing for Singularly Perturbed Differential Equations*

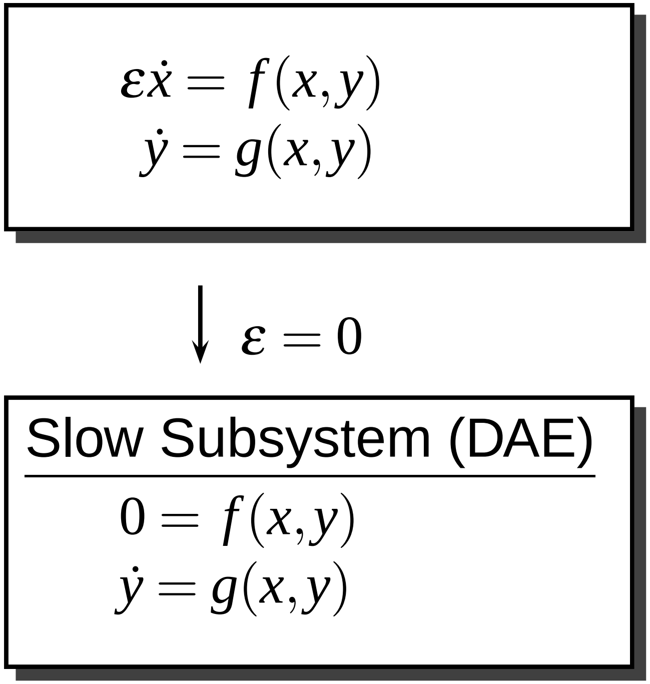
Acknowledgments

- NSF Grant DMS-0514468: *RUI: Reduced System Computing for Singularly Perturbed Differential Equations*
- Undergraduate students:
  - *Brian Kinney*
  - *Tomas Gruszka*
  - *Dimitar Simeonov*

Slow and Fast Subsystems, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

$$\boxed{\begin{aligned} \varepsilon\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}}$$

Slow and Fast Subsystems, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$



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$$\downarrow \; \varepsilon = 0$$

Slow Subsystem (DAE)

$$\begin{aligned} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

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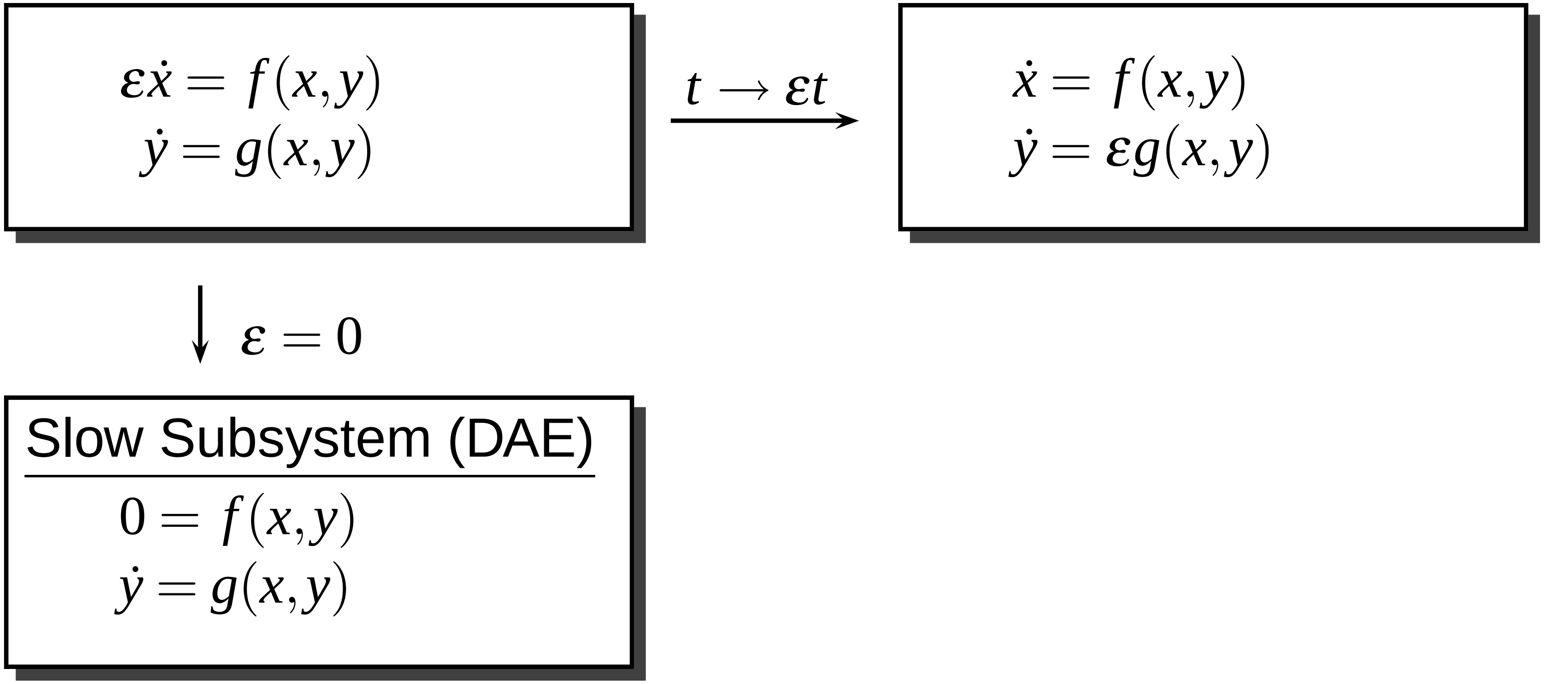

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$t \longrightarrow \varepsilon t$

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= \varepsilon g(x,y) \end{aligned}$$

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\boxed{\begin{array}{l} \text{Slow Subsystem (DAE)} \\ \hline 0 = f(x,y) \\ \dot{y} = g(x,y) \end{array}} & \boxed{\begin{array}{l} \text{Fast Subsystem} \\ \hline \dot{x} = f(x,y) \\ \dot{y} = 0 \end{array}}
\end{tikzcd}$$

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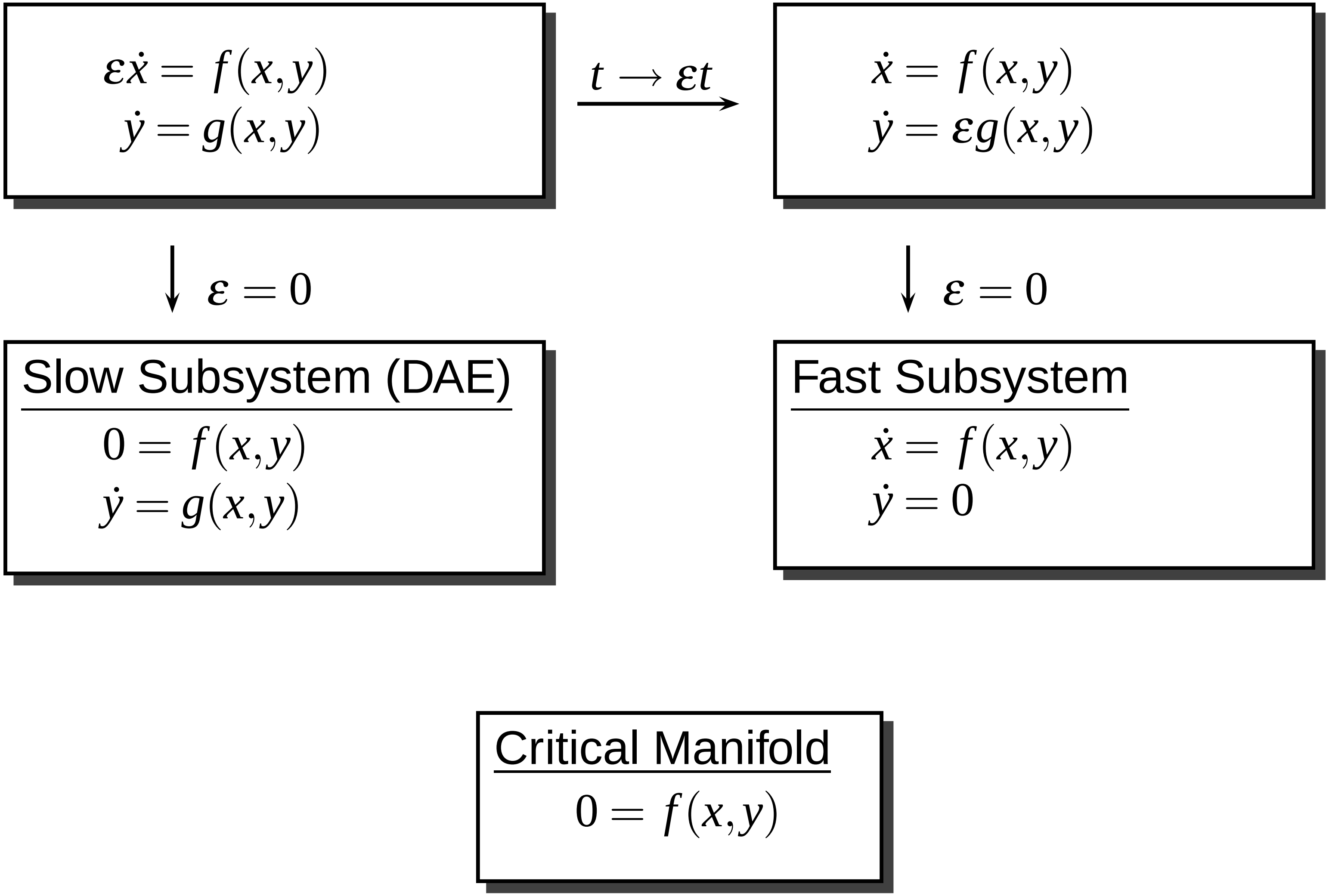

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Fast Subsystem

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Critical Manifold

$$0 = f(x,y)$$

Reduced System Computing

Goal:

❑ Create computational tools for the study of the reduced system of a singularly perturbed differential equation.

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Complications:

- Canards
- Fast periodic orbits (and more general ω limit sets of the fast subsystem)

Slow Subsystem - A Few More Details

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Differentiate $0 = f(x,y)$, solve for $\dot{x}$ to obtain $\dot{x} = -(D_x f)^{-1}(D_y f)g(x,y)$. Multiply by $\det(D_x f)$.

Desingularized slow equations:

$$\begin{aligned} \dot{x} &= -(\operatorname{adj} D_x f)(D_y f)g(x,y) \\ \dot{y} &= \det(D_x f)g(x,y) \end{aligned}$$

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Fold points:

$$f(x,y) = 0, \quad \det(D_x f) = 0$$

Fold points are saddle-node equilibria of the fast subsystem.

Example: Two Coupled Neurons

$$
\begin{aligned}
\dot{v}_1 &= -v_1 + \tanh(\sigma_1 v_1) - q_1 - \omega f(v_2)(v_1 + 4) \\
\dot{v}_2 &= -v_2 + \tanh(\sigma_2 v_2) - q_2 - \omega f(v_1)(v_2 + 4) \\
\dot{q}_1 &= \varepsilon(-q_1 + v_1) \\
\dot{q}_2 &= \varepsilon(-q_2 + v_2)
\end{aligned}
$$

$$f(v) = \frac{1}{1 + e^{-40(v - 1/75)}}$$

- Each (v_i, q_i) is a relaxation oscillator.
- When one is firing, the other's v nullcline is depressed ("reciprocal inhibition").
- This system has two fast variables and a two dimensional critical manifold.

Fast/Slow System Definition File: `cpldosc.fs`

```
# Definitions for the coupled oscillator fast/slow system.
cpldosc
# Fast variables:  v1, v2
2
v1
v2
# Slow variables:  q1, q2
2
q1
q2
# Parameters:
3
omega
sigma1
sigma2
# Vector field for the fast variables
-v1+tanh(sigma1*v1) - q1 - omega*(v1+4)/(1+exp(-40*(v2-1/75)))
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Input file: `cpldosc.fs`

Output files:

`fs_cpldosc.c`	General C functions
`fs_cpldosc_cvode.c`	C functions for CVODE
`fs_cpldosc_ida.c`	C functions for IDA
`fs_cpldosc.m`	MATLAB functions

Computer Code Generated

The generated code includes:

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The generated code includes:

- *Fast vector field and IVP solver*

Fast vector field, to be used with CVODE

```
int cpldoscF_cv(realtype t, N_Vector Xvec, N_Vector Xdotvec,
                                                      void *params)
```

Fast vector field Jacobian

```
int cpldoscFx_cv(long int N, DenseMat Fx, realtype t,
                        N_Vector Xvec, N_Vector Fvec, void *params,
                       N_Vector tmp1, N_Vector tmp2, N_Vector tmp3)
```

Solve the fast subsystem IVP

```
void cpldosc_fast(FILE *fastfile, double X[], double Y[],
                      double params[], SolverParams *solver_params)
```

Computer Code Generated

The generated code includes:

- *Slow subsystem DAE, configured for IDA*

  Compute residuals for the IDA DAE solver

  ```
  int cpldosc_idares(realtype t, N_Vector Zvec, N_Vector Zdotvec,
                                         N_Vector rvec, void *params)
  ```

  Jacobian for IDA DAE solver

  ```
  int cpldosc_idajac(long int Neq, realtype t, N_Vector Zvec,
                                     N_Vector Zdotvec, N_Vector rvec,
                         realtype c_j, void *params, DenseMat jacmat,
                         N_Vector tmp1, N_Vector tmp2, N_Vector tmp3)
  ```

  Solve the slow subsystem DAE IVP

  ```
  void cpldosc_slow_dae(FILE *slowfile, double X[], double Y[],
                        double params[], SolverParams *solver_params)
  ```

Computer Code Generated

The generated code includes:

- ❑ *Desingularized slow subsystem (for CVODE)*

  Desingularized slow subsystem vector field

```
int cpldosc_SlowDE(realtype t, N_Vector Zvec, N_Vector Zdotvec,
                                                        void *params)
```

  Desingularized slow subsystem IVP solver

```
void cpldosc_slow_des(FILE *slowfile, double X[], double Y[],
                      double params[], SolverParams *solver_params)
```

Computer Code Generated

The generated code includes:

- *Fold function and Jacobian*

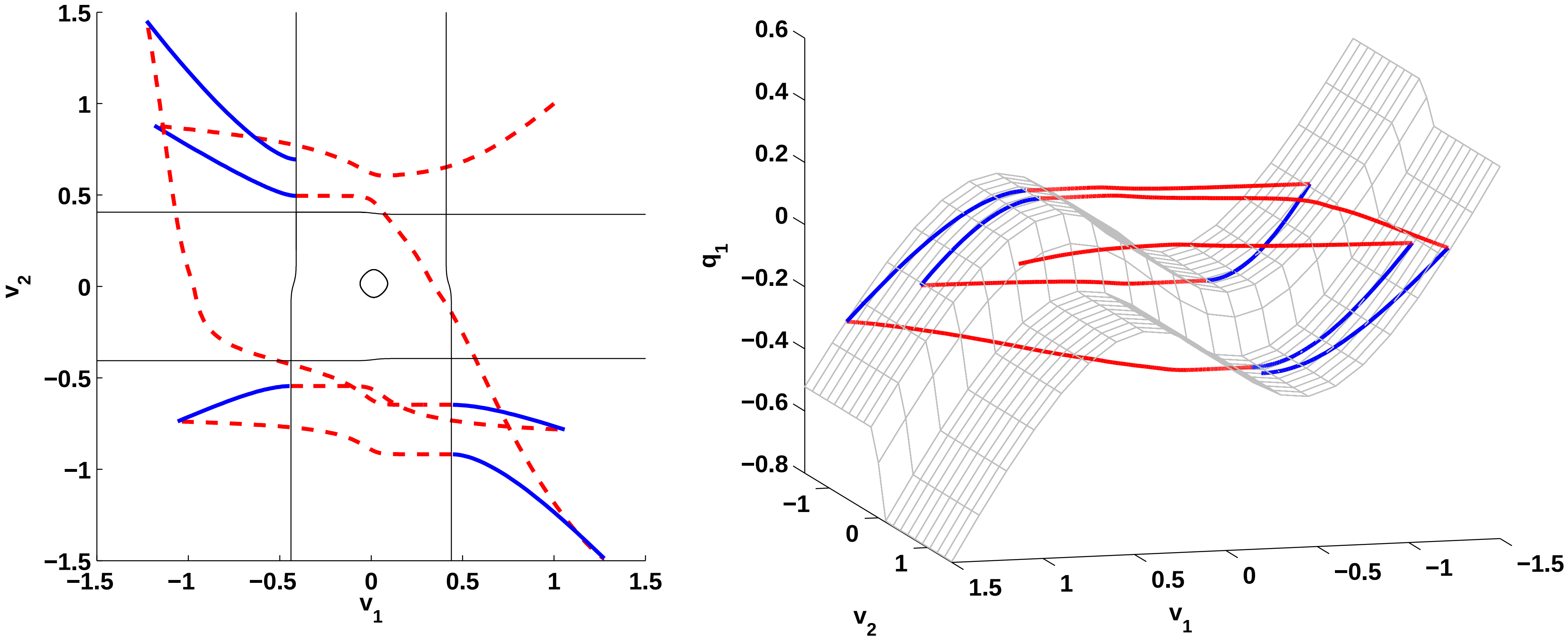
  Fold function

  ```
  realtype cpldosc_foldfunc_nv(N_Vector Zvec, void *params)
  ```

  Fold function gradient (with respect to all variables)

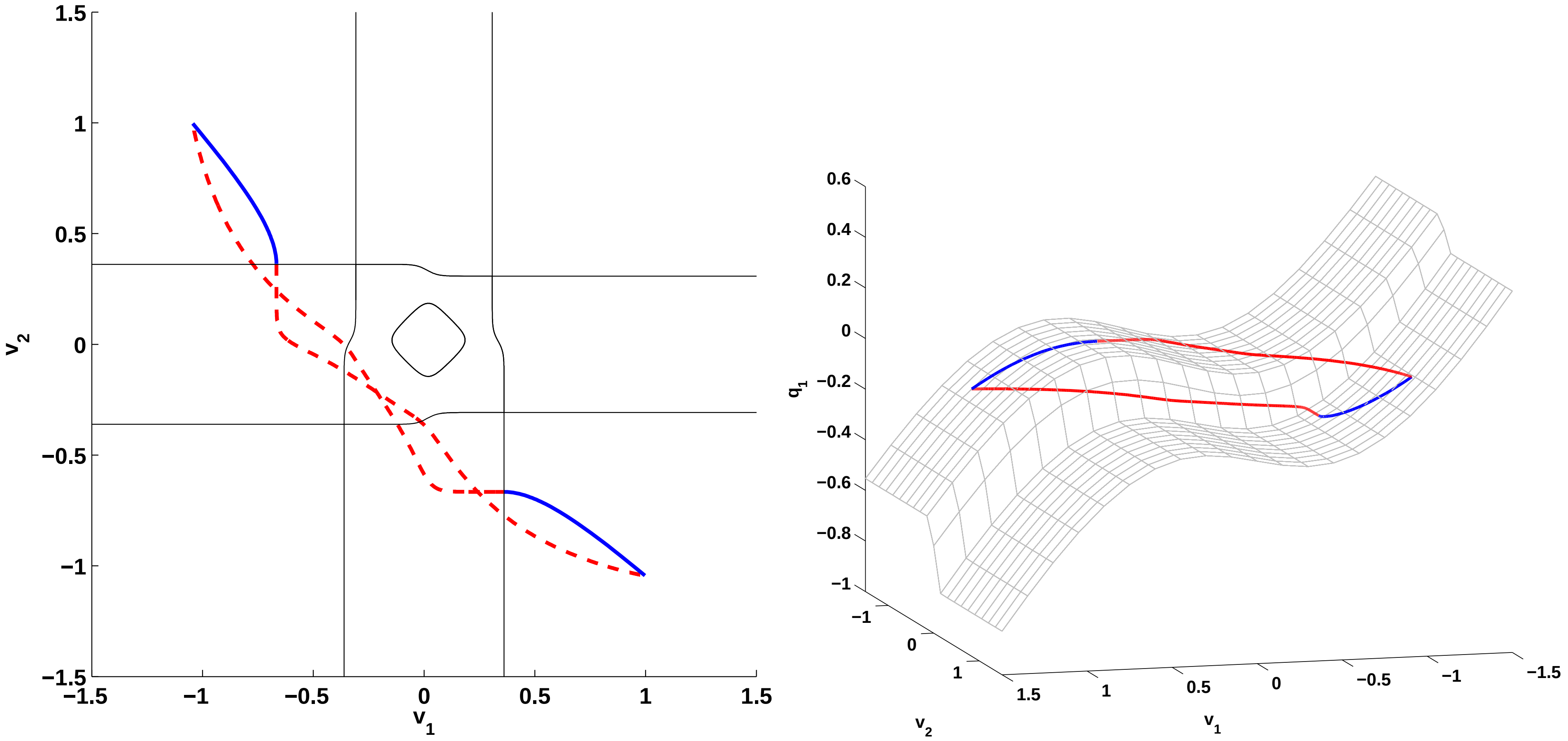
  ```
  void cpldosc_foldfunc_grad_nv(double *grad, N_Vector Zvec,
                                                       void *params)
  ```

Example: Initial Value Problem for the Coupled Oscillator System



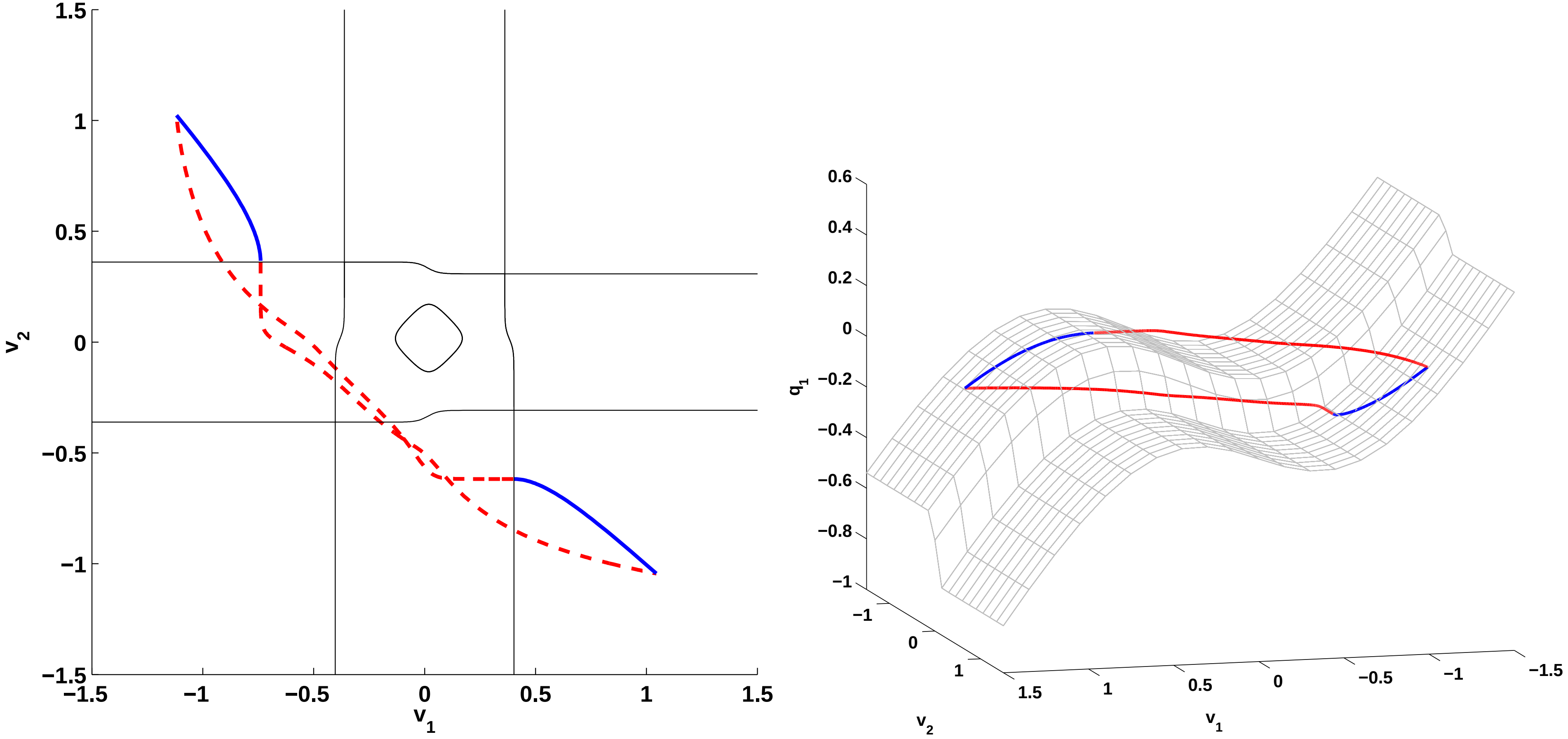
$(\omega = 0.05,\ \sigma_1 = 1.5,\ \sigma_2 = 2.6)$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



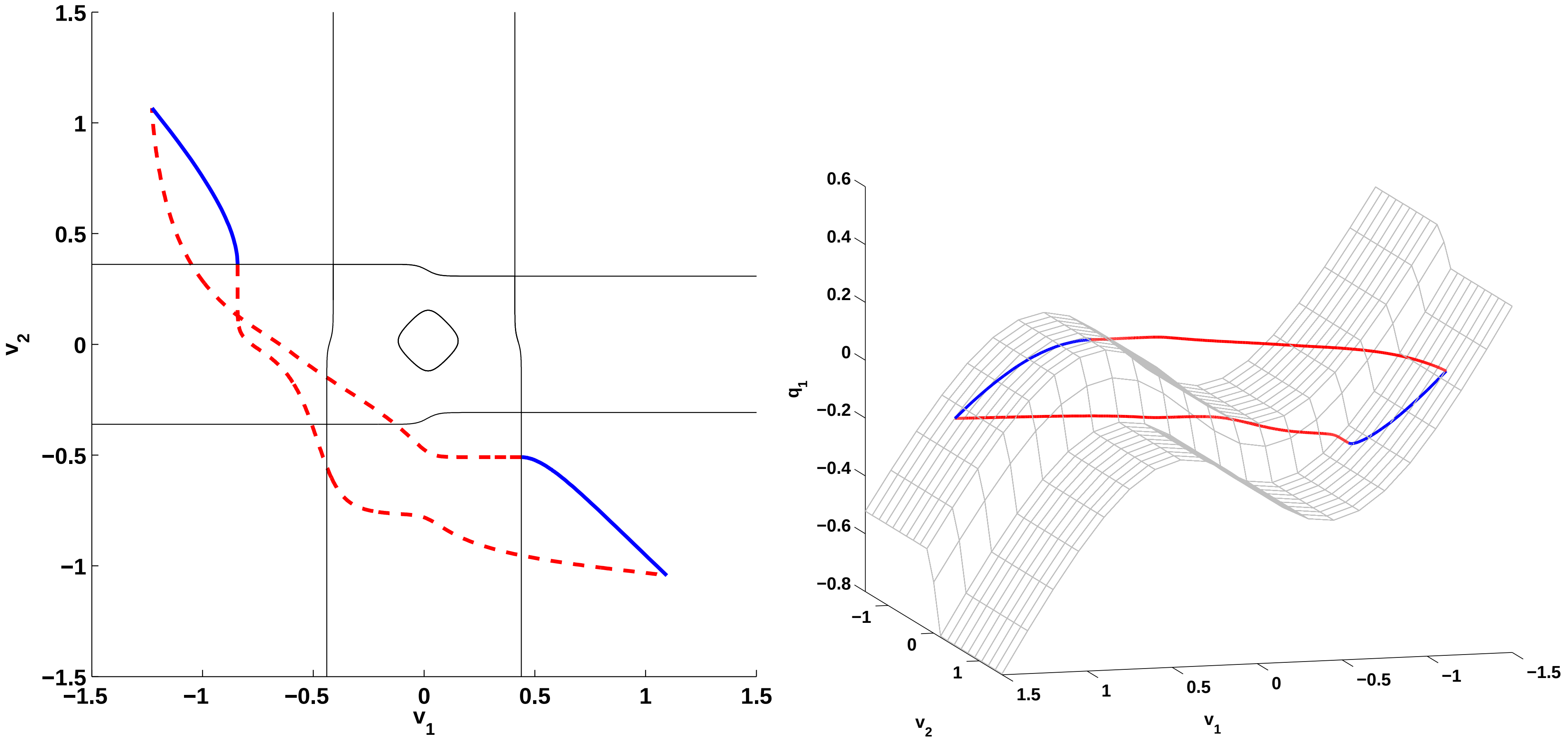
$\sigma_1 = 1.2$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



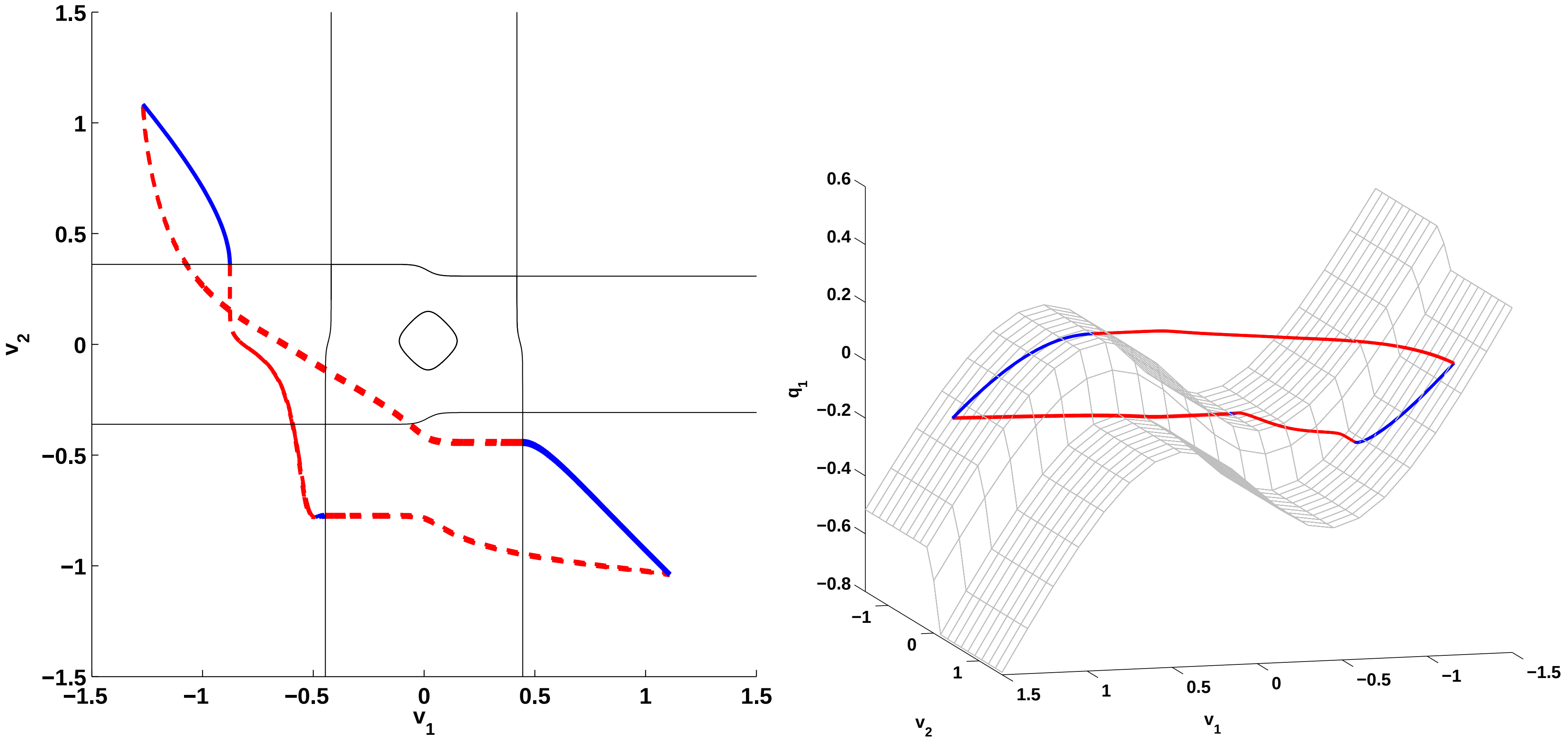
$$\sigma_1 = 1.3$$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



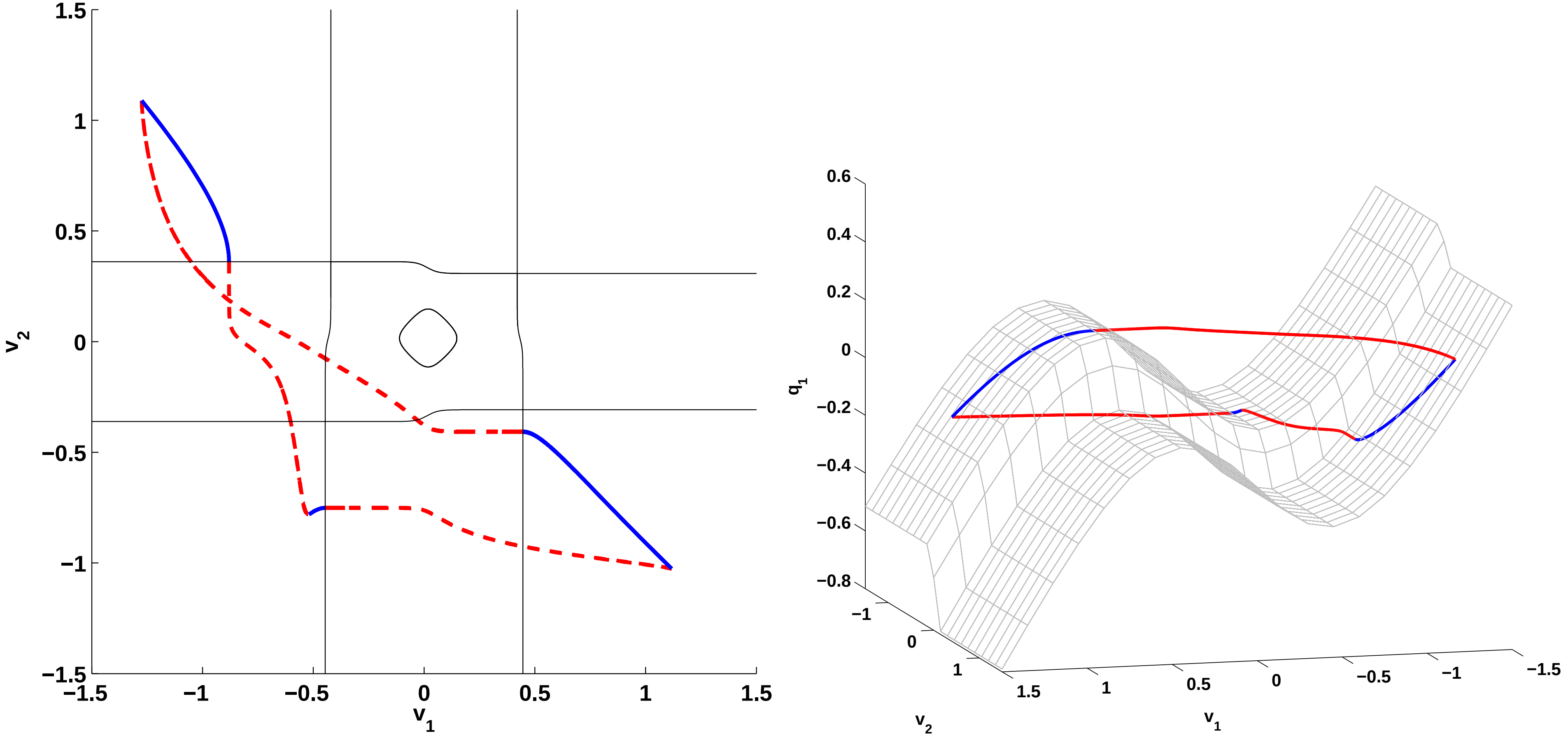
$\sigma_1 = 1.4$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



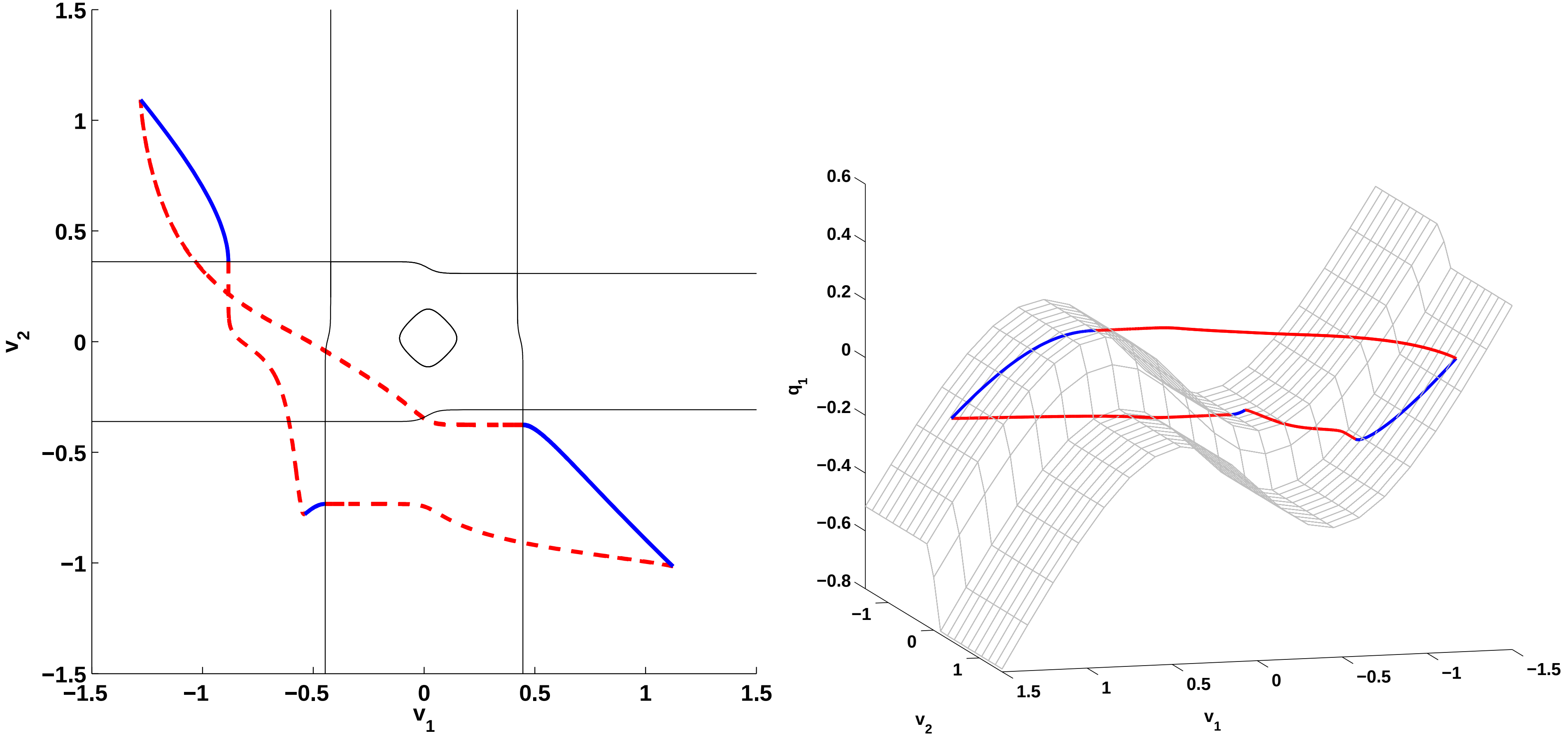
$$\sigma_1 = 1.5925$$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



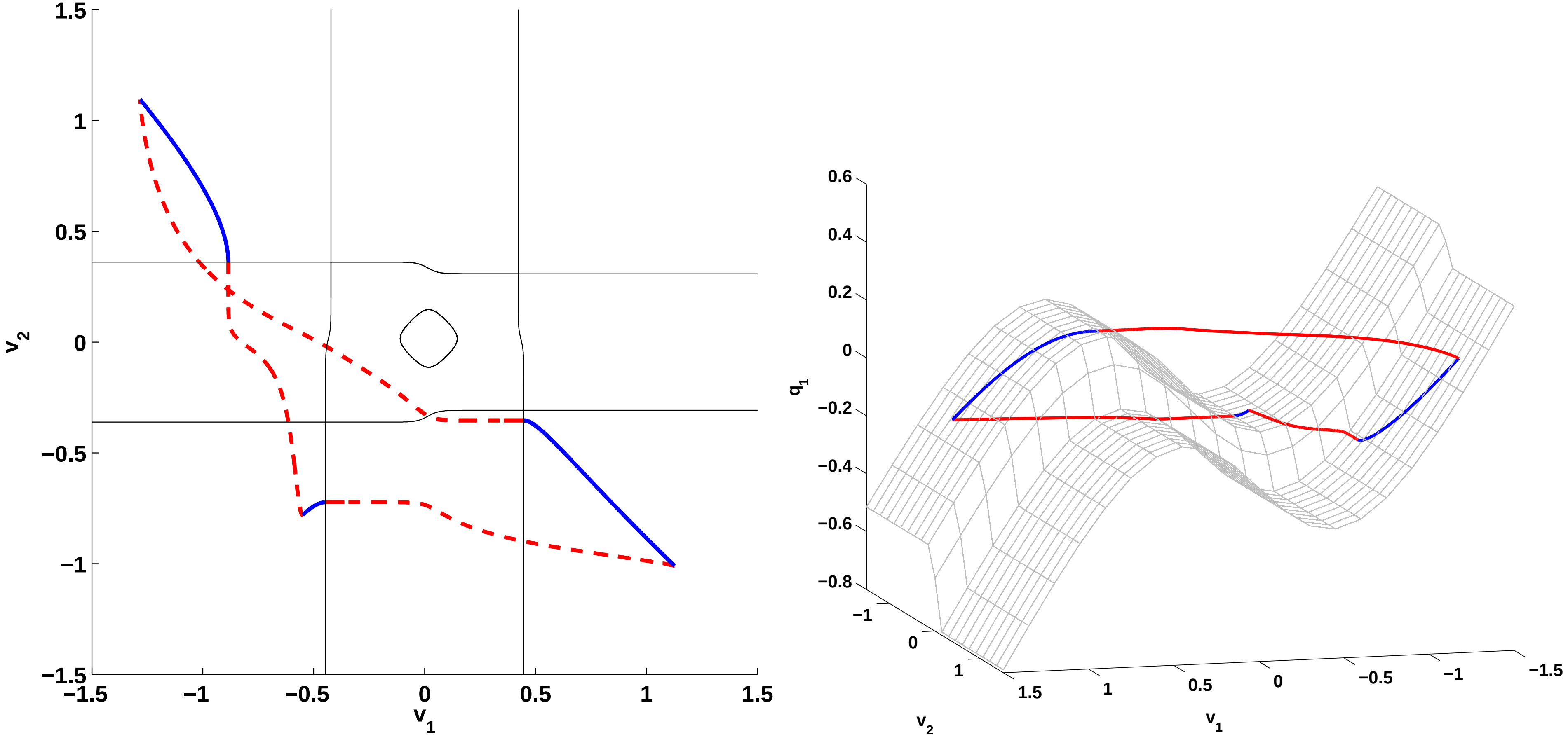
$$\sigma_1 = 1.6125$$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



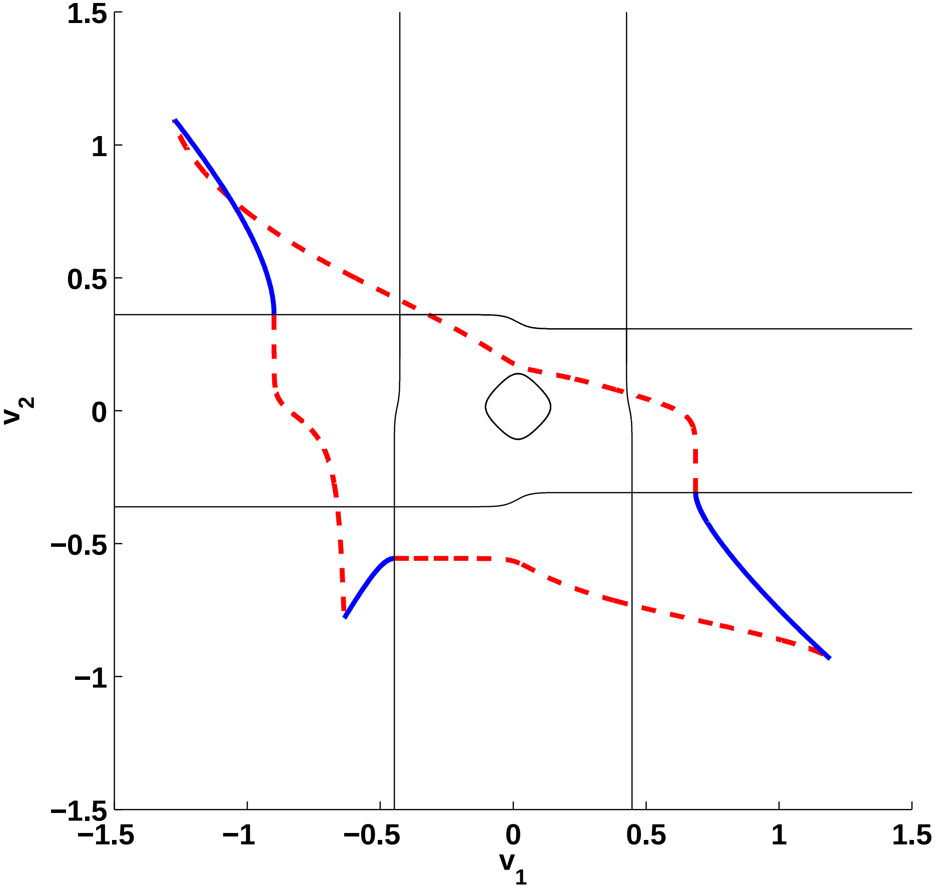
$\sigma_1 = 1.6250$

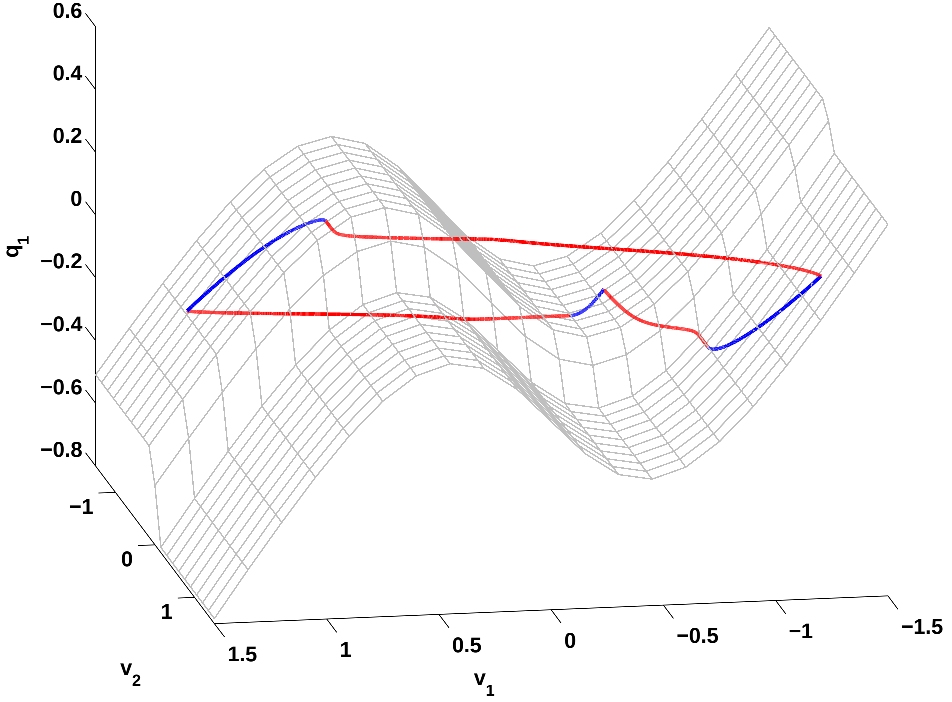
Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



$$\sigma_1 = 1.6325$$

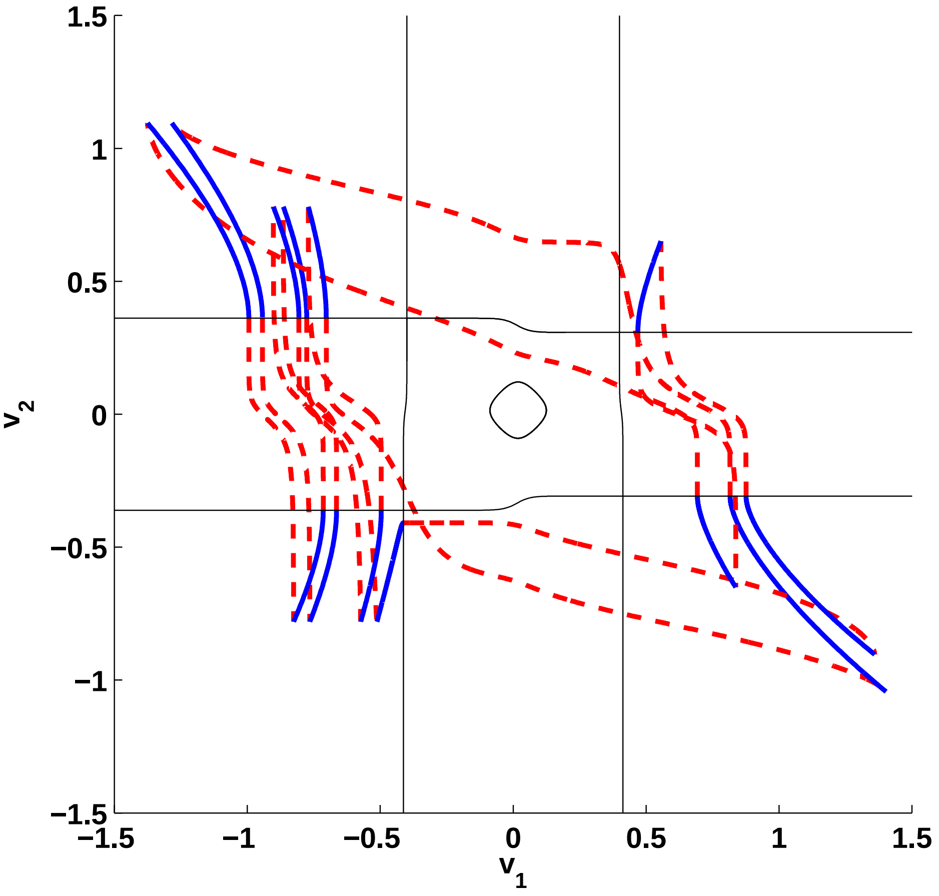
Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)

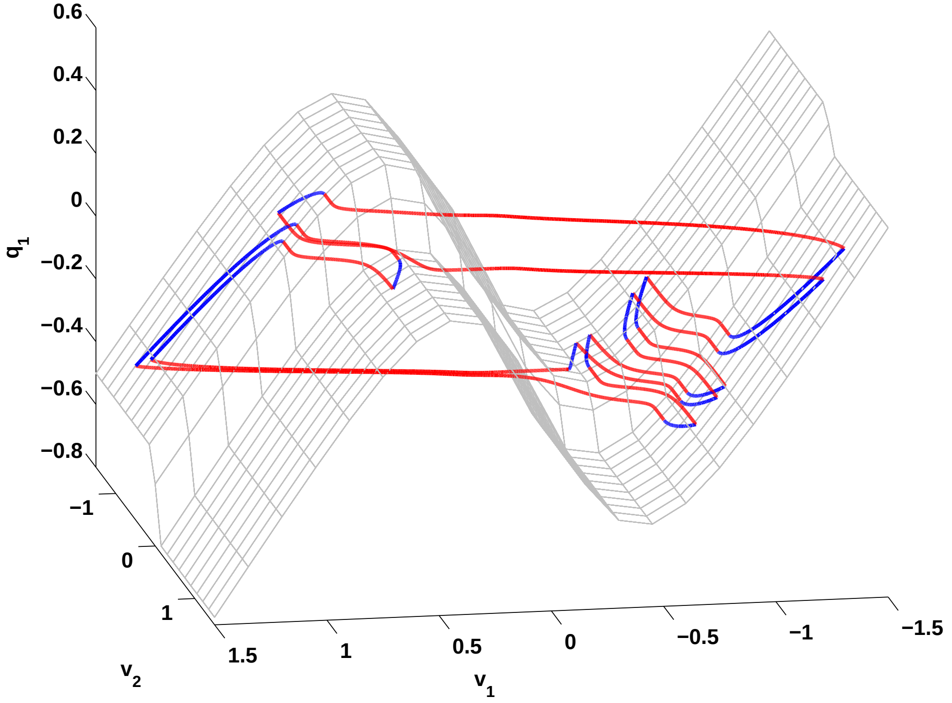




$\sigma_1 = 1.8$

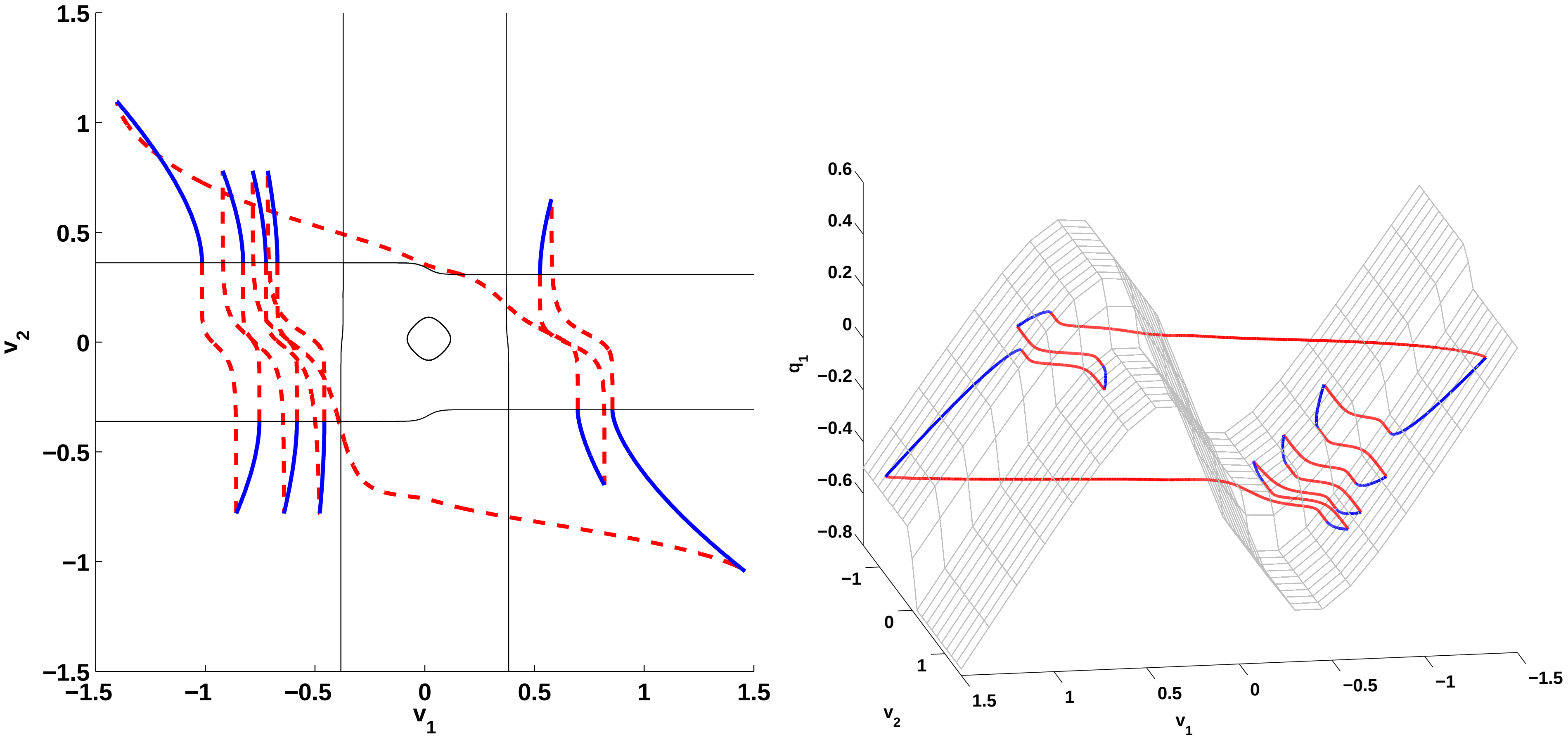
Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)





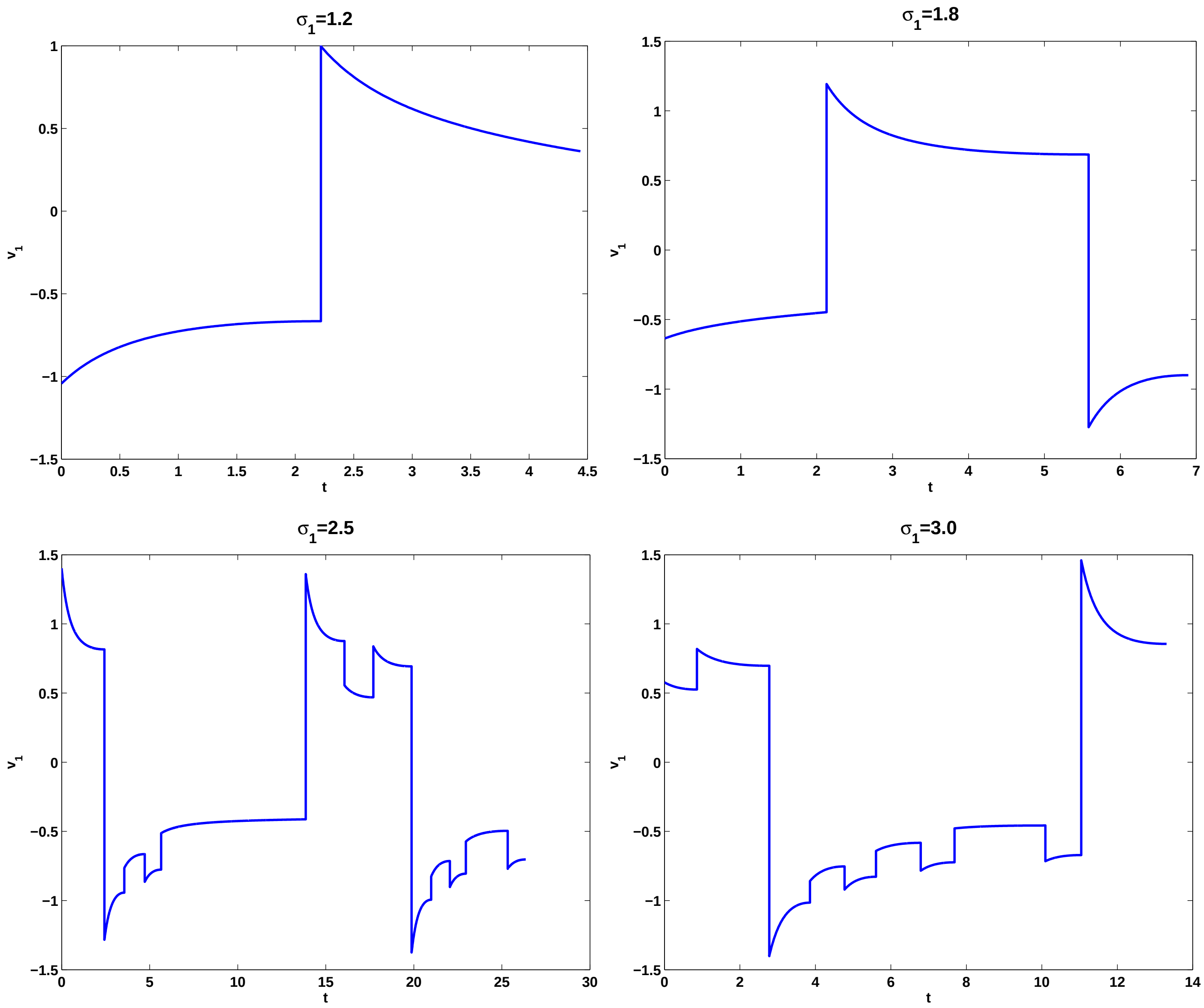
$\sigma_1 = 2.5$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



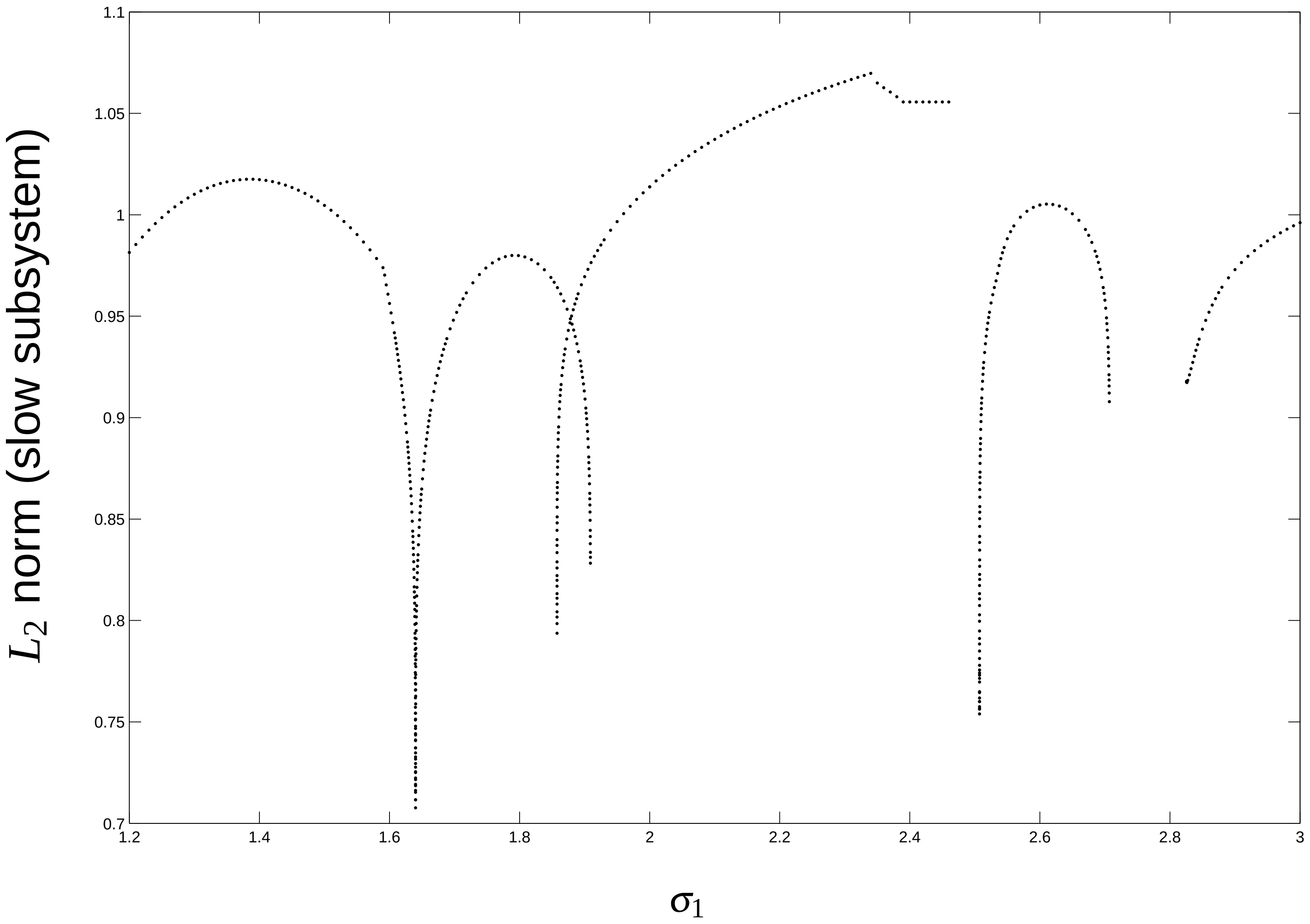
$\sigma_1 = 3$

Periodic Orbits of the Coupled Oscillator System $(\omega = 0.05,\ \sigma_2 = 1.2)$



Bifurcation Diagram: Stable Periodic Orbits

(“Poor man’s” continuation; not a complete diagram)



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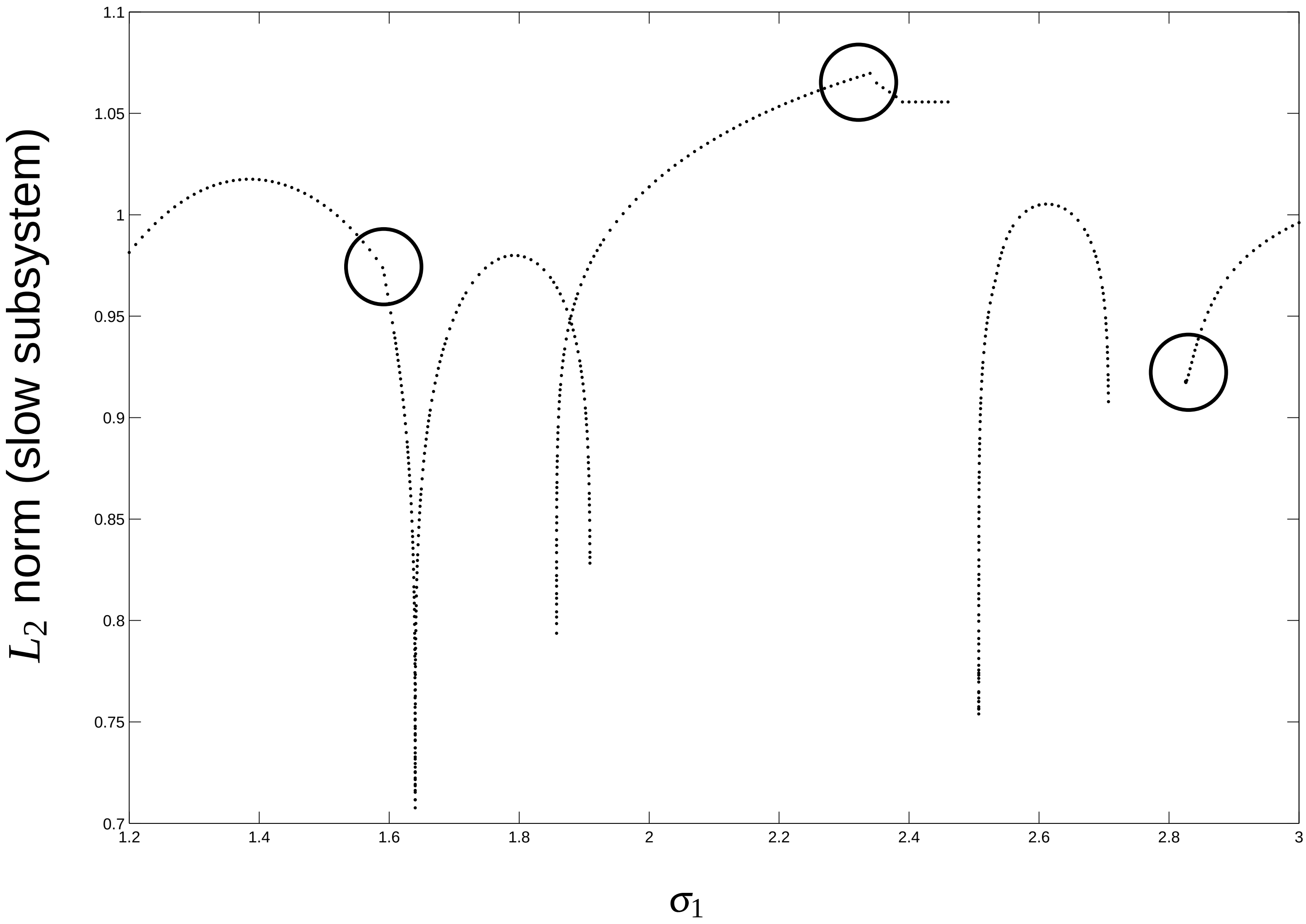

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- ❏ (*Fold-to-fold examples where the slow flow is towards the fold or away from the fold have been observed.*)
- ❏ Periodic orbits homoclinic to a folded saddle.

The investigation is not complete!

Current and Future Development

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- Computing and continuing periodic orbits.

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- Computing and continuing periodic orbits.
- Find fast/slow orbit homoclinic to folded singular points.
  These play an import role in the bifurcations of periodic orbits:
  - J. Guckenheimer, K. Hoffman, W. Weckesser,
    Bifurcations of relaxation oscillations near folded saddles,
    Int. J. Bif. Chaos 15(11), (2005)
  - M. Brøns, M. Krupa, M. Wechselberger,
    Mixed Mode Oscillations Due to the Generalized Canard Phenomenon,
    Fields Institute Communications Vol. 49, 39-63 (2006)