Reduced System Computing

for

Singularly Perturbed Differential Equations

Warren Weckesser

Department of Mathematics
Colgate University

Acknowledgments

- NSF Grant DMS-0514468: *RUI: Reduced System Computing for Singularly Perturbed Differential Equations*

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- Undergraduate students:
  - *Brian Kinney*
  - *Tomas Gruszka*
  - *Dimitar Simeonov*

Slow and Fast Subsystems, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

$$\boxed{\begin{aligned} \varepsilon\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}}$$

Slow and Fast Subsystems, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

$$\begin{aligned} \varepsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

$$\downarrow \; \varepsilon = 0$$

Slow Subsystem (DAE)

$$\begin{aligned} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned}$$

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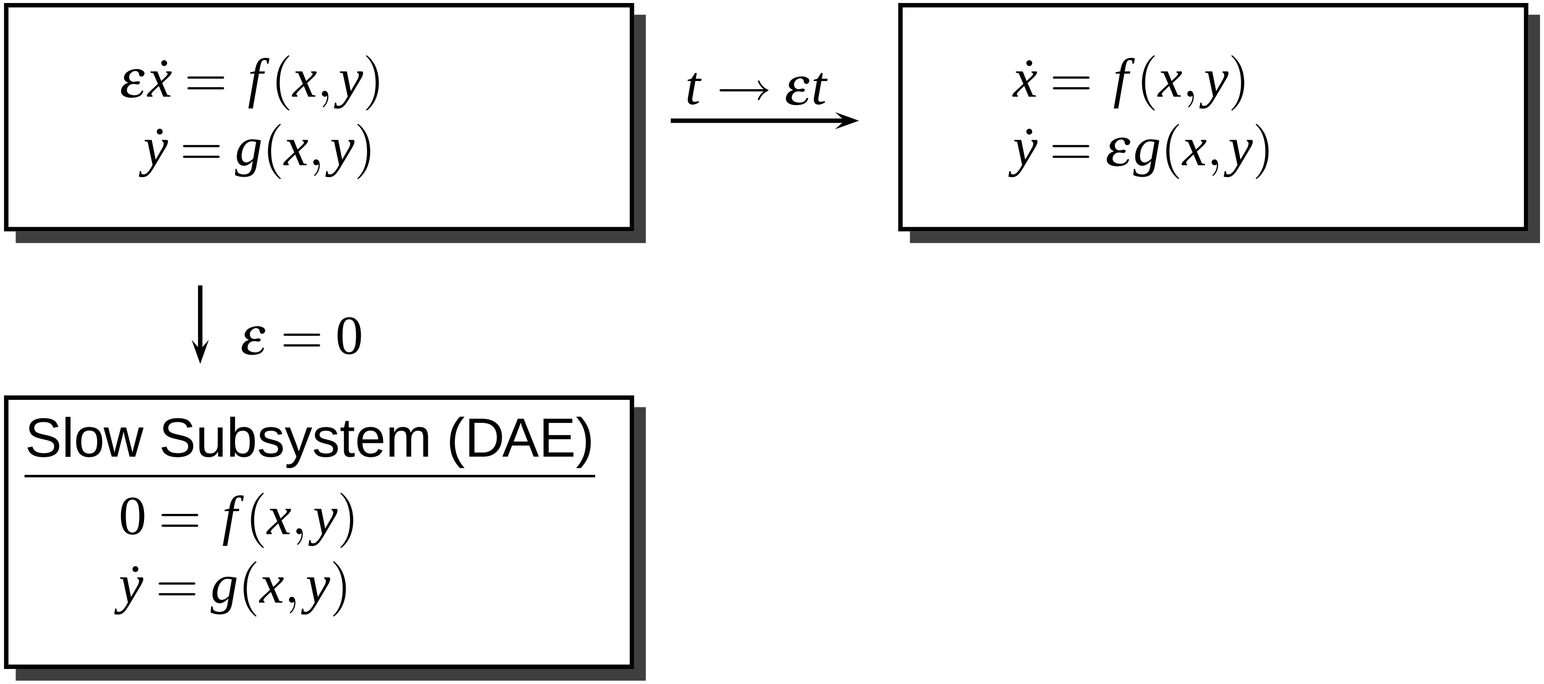

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$\xrightarrow{t \longrightarrow \varepsilon t}$

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$$\begin{tikzcd}
\boxed{\begin{array}{l} \varepsilon\dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{array}} \arrow[r, "t \longrightarrow \varepsilon t"] \arrow[d, "\varepsilon = 0"] & \boxed{\begin{array}{l} \dot{x} = f(x,y) \\ \dot{y} = \varepsilon g(x,y) \end{array}} \arrow[d, "\varepsilon = 0"] \\
\boxed{\begin{array}{l} \text{Slow Subsystem (DAE)} \\ \hline 0 = f(x,y) \\ \dot{y} = g(x,y) \end{array}} & \boxed{\begin{array}{l} \text{Fast Subsystem} \\ \hline \dot{x} = f(x,y) \\ \dot{y} = 0 \end{array}}
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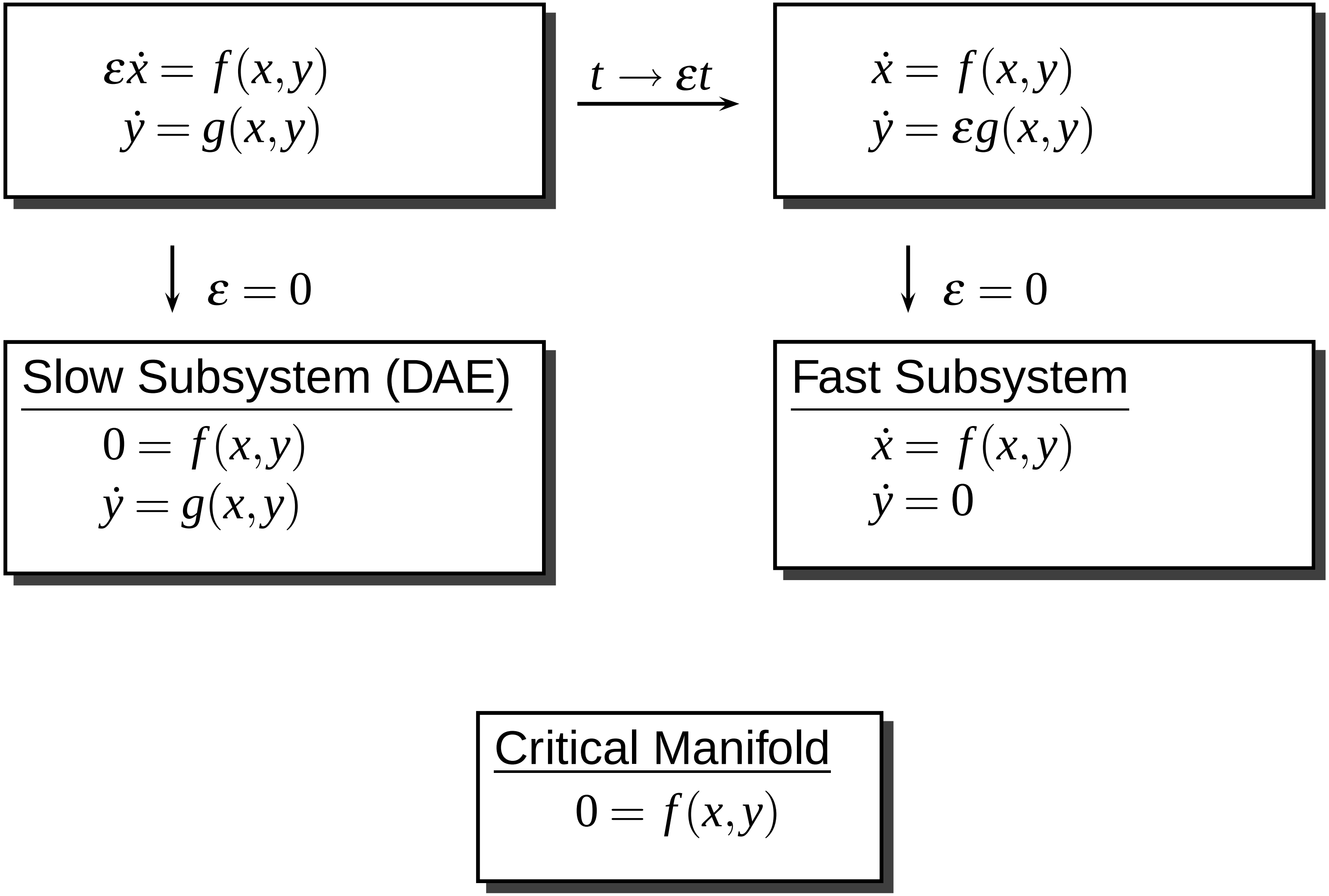
Slow and Fast Subsystems, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$



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Fast Subsystem

$$\begin{aligned} \dot{x} &= f(x,y) \\ \dot{y} &= 0 \end{aligned}$$

Critical Manifold

$$0 = f(x,y)$$

Example: The Periodically Forced van der Pol System

$$x'' + d(x^2 - 1)x' + x = a\sin(\nu\tau)$$

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New parameters: $\varepsilon = 1/d^2$, $\omega = \dfrac{\nu d}{2\pi}$

New variables: $t = \sqrt{\varepsilon}\tau$, $\theta = \omega t$, $y = \varepsilon\dot{x} + x^3/3 - x$

Then

$$\begin{aligned} \varepsilon\dot{x} &= x - \tfrac{1}{3}x^3 + y \\ \dot{y} &= -x + a\sin(2\pi\theta) \\ \dot{\theta} &= \omega \end{aligned}$$

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Symmetry: $x \longrightarrow -x$, $y \longrightarrow -y$, $\theta \longrightarrow \theta + 1/2$

Forced van der Pol Slow and Fast Subsystems

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$\Big\downarrow$ Eliminate y and desingularize

$$\begin{aligned}
\dot{\theta} &= \omega(x^2 - 1)\\
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(Time reversed for $|x| < 1$.)

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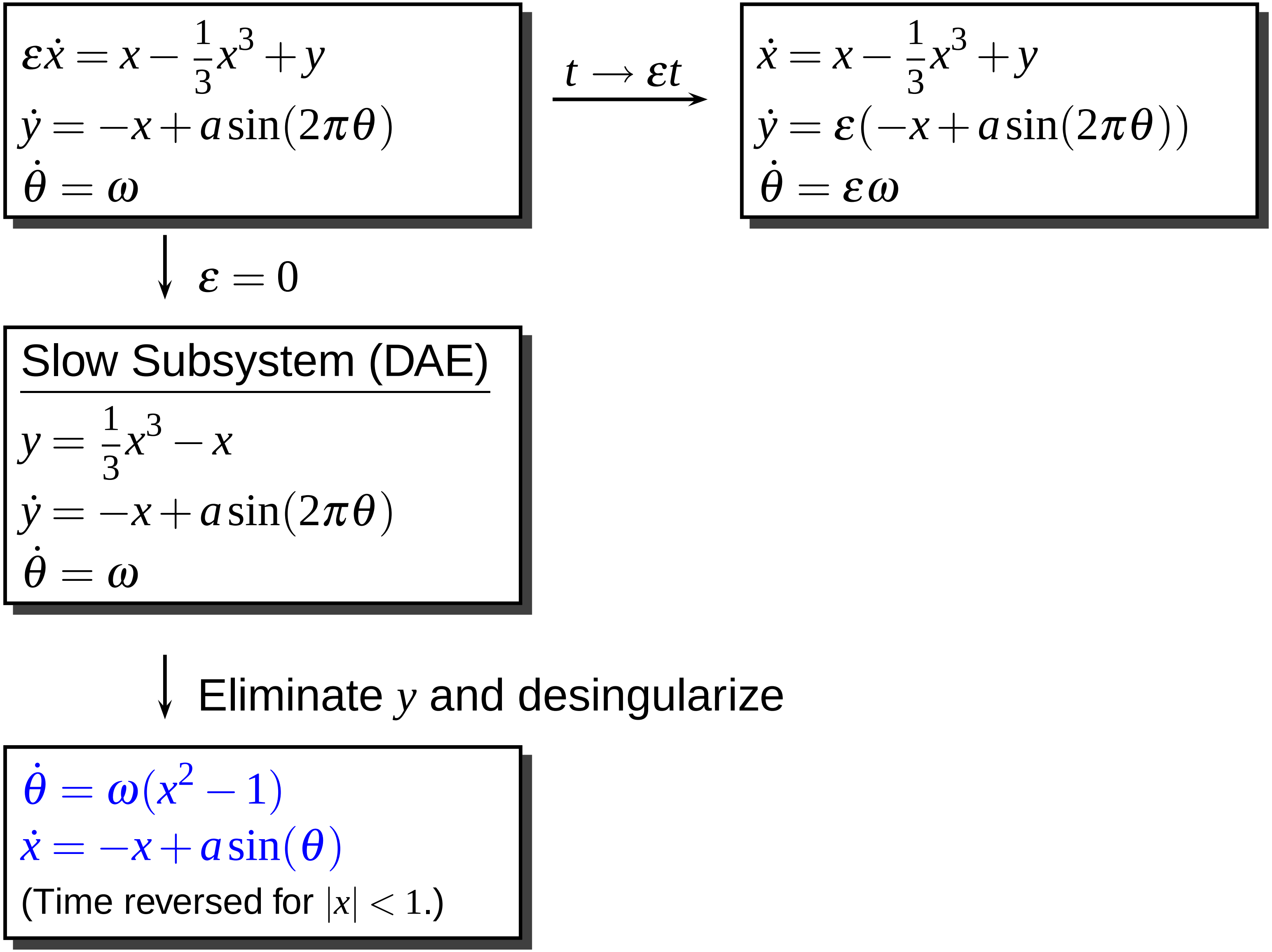

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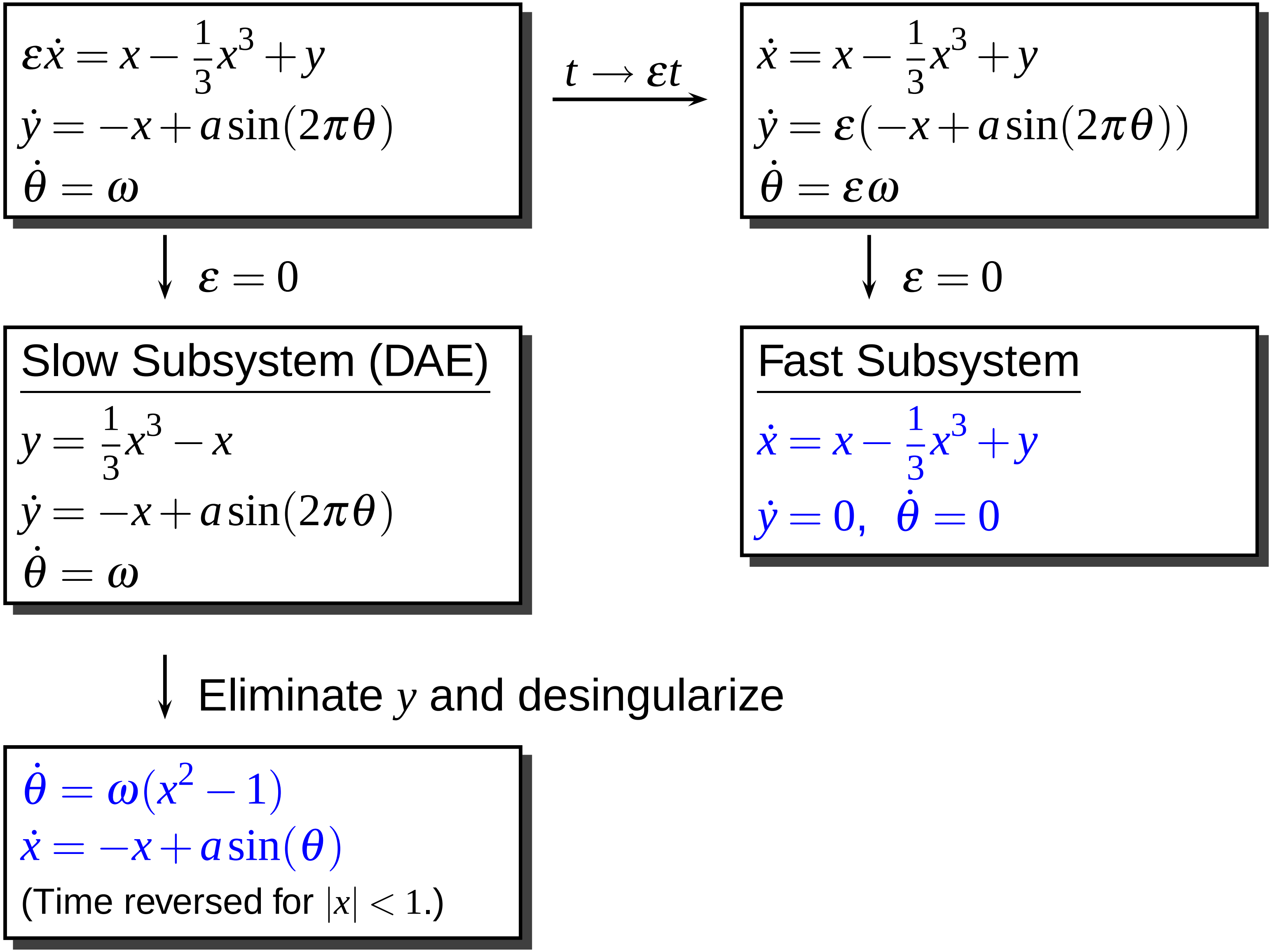

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(Time reversed for $|x| < 1$.)

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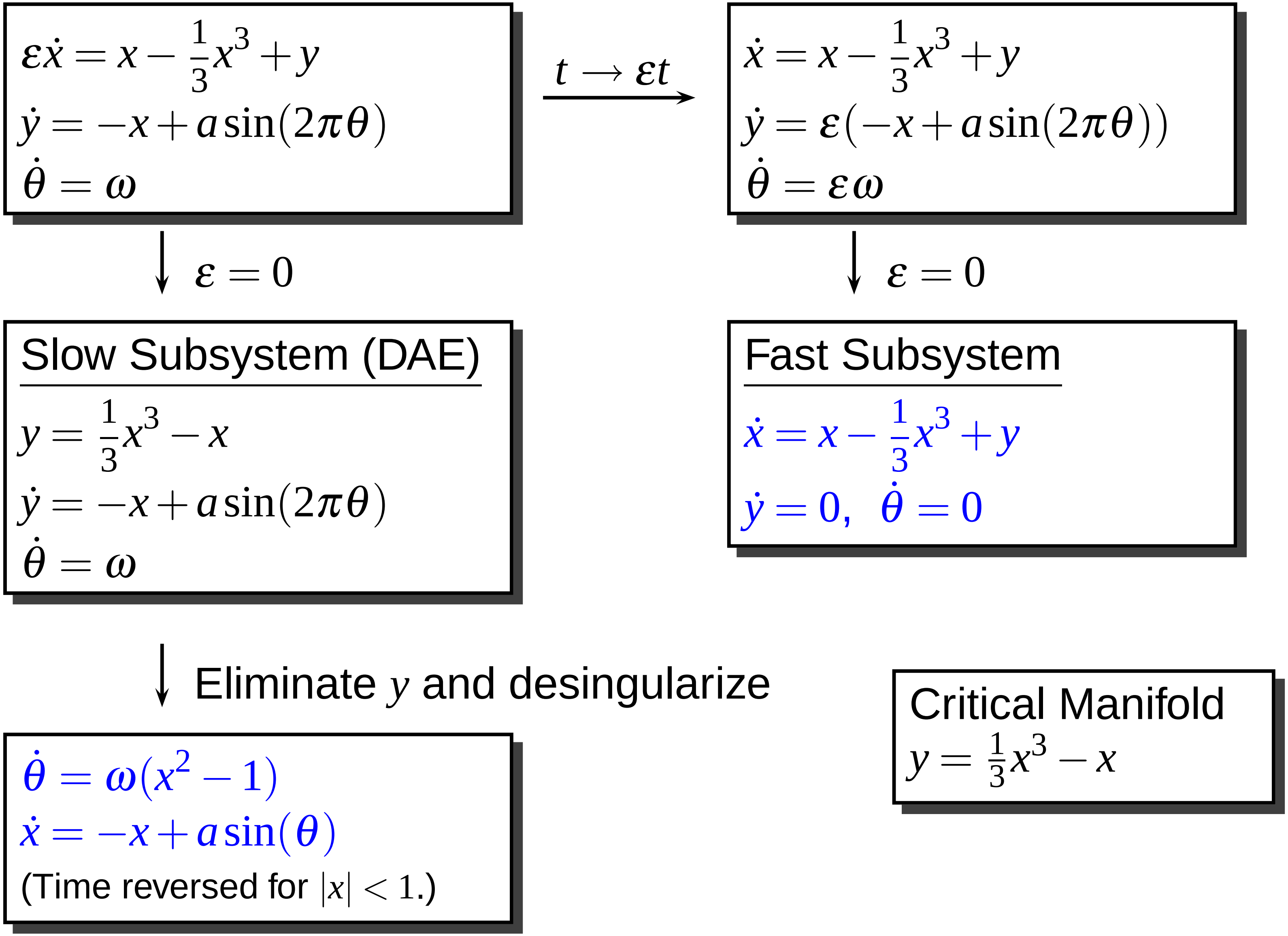

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Critical Manifold

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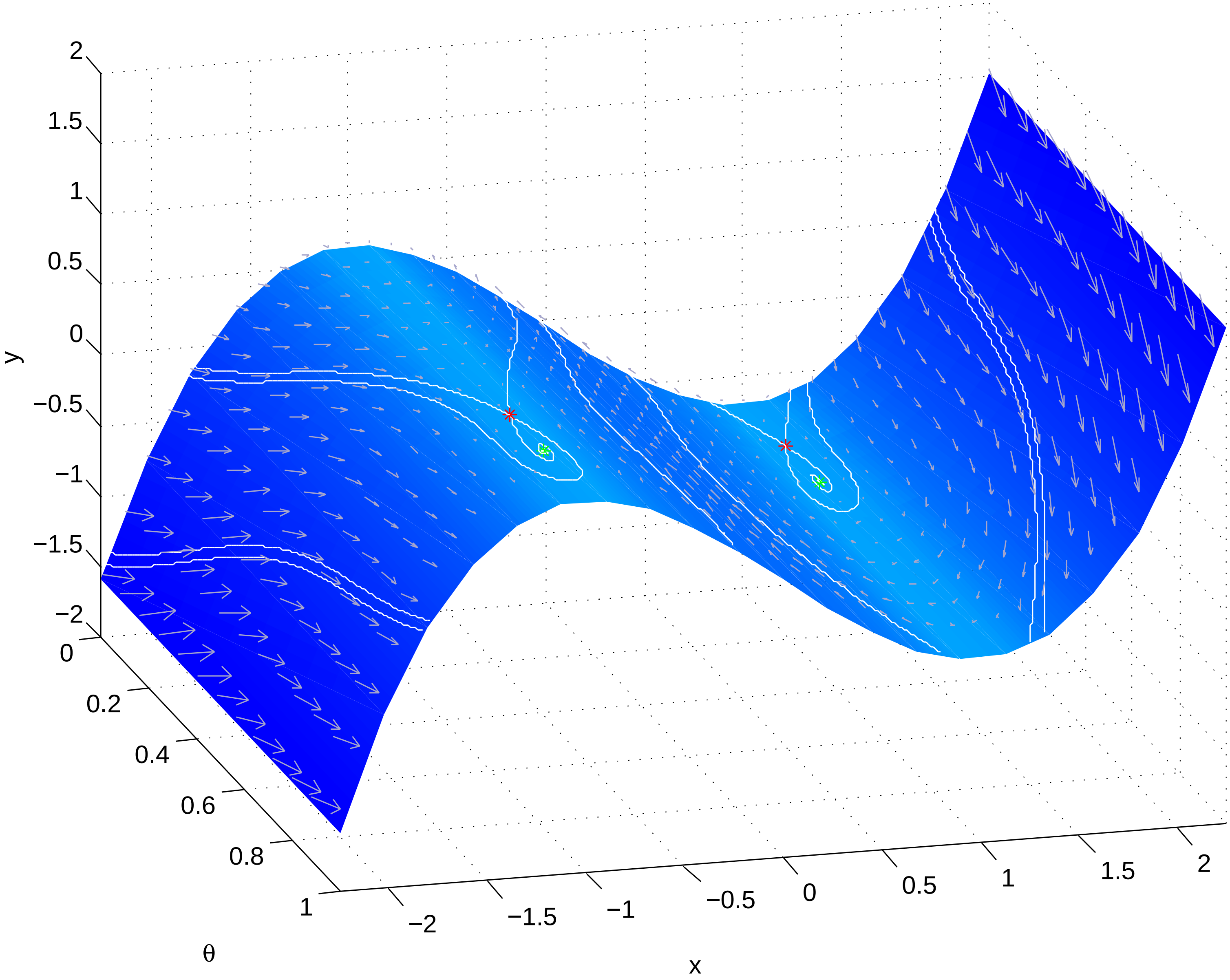
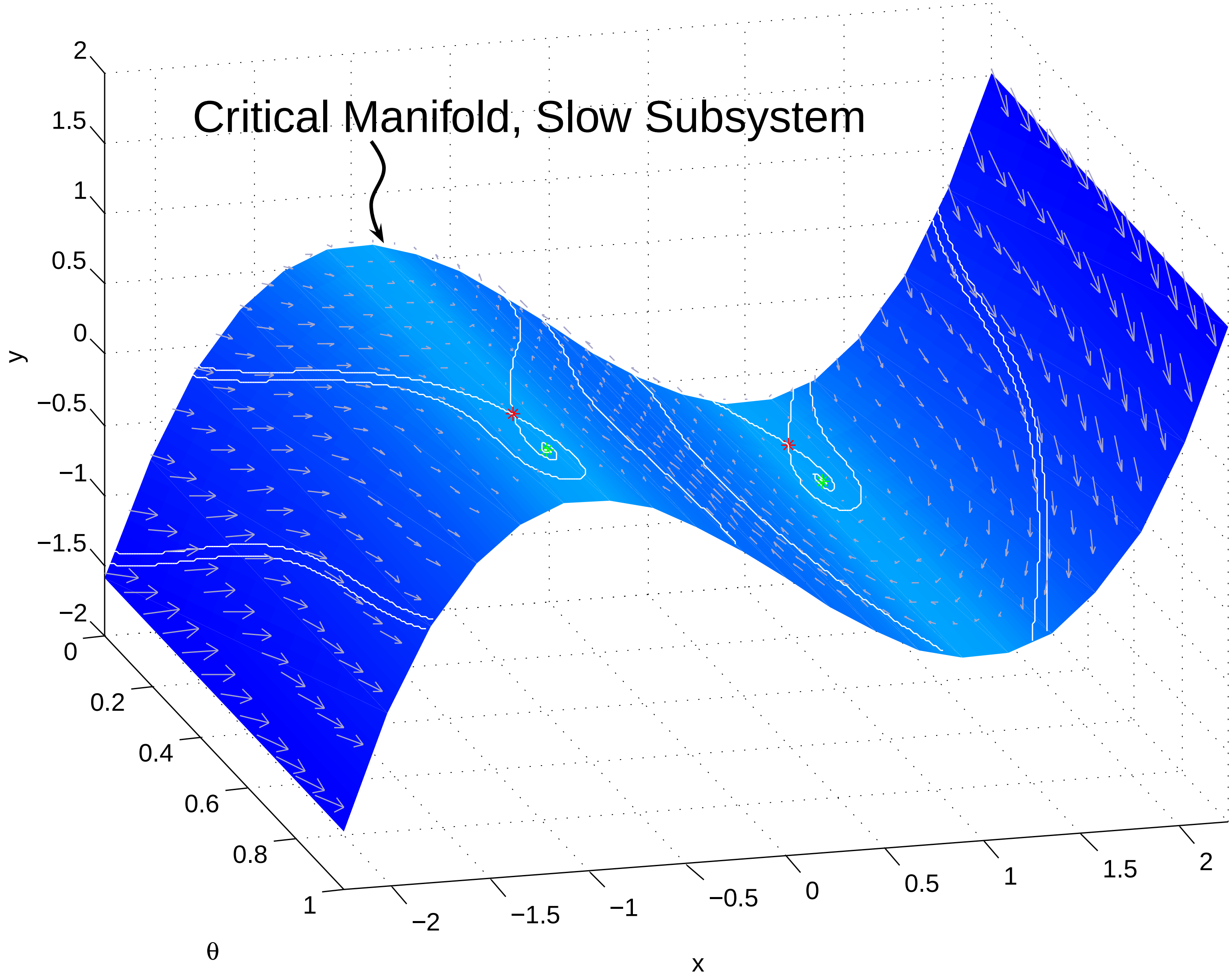
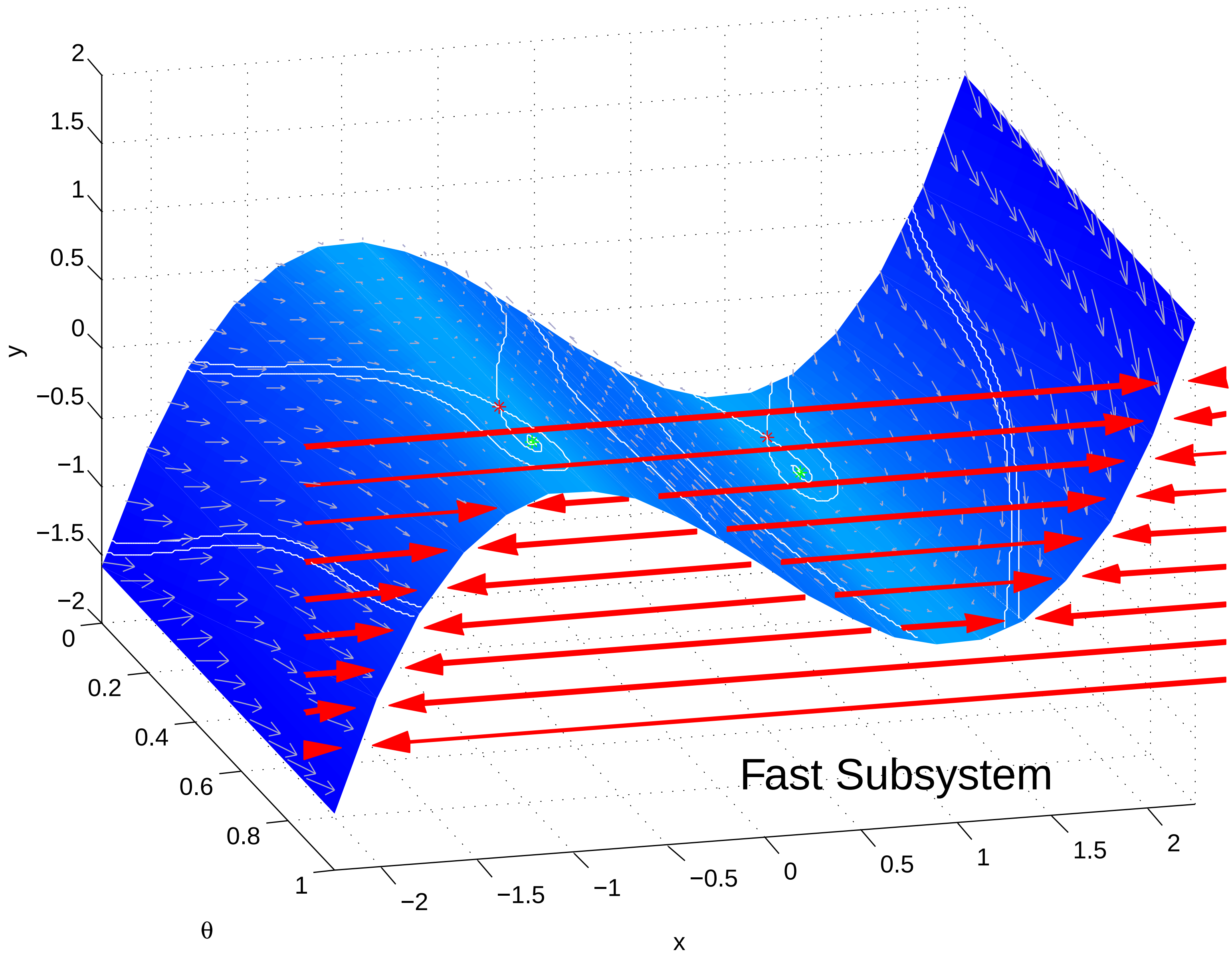
Forced van der Pol: Fast and Slow Subsystems



Forced van der Pol: Fast and Slow Subsystems


Critical Manifold, Slow Subsystem

Forced van der Pol: Fast and Slow Subsystems



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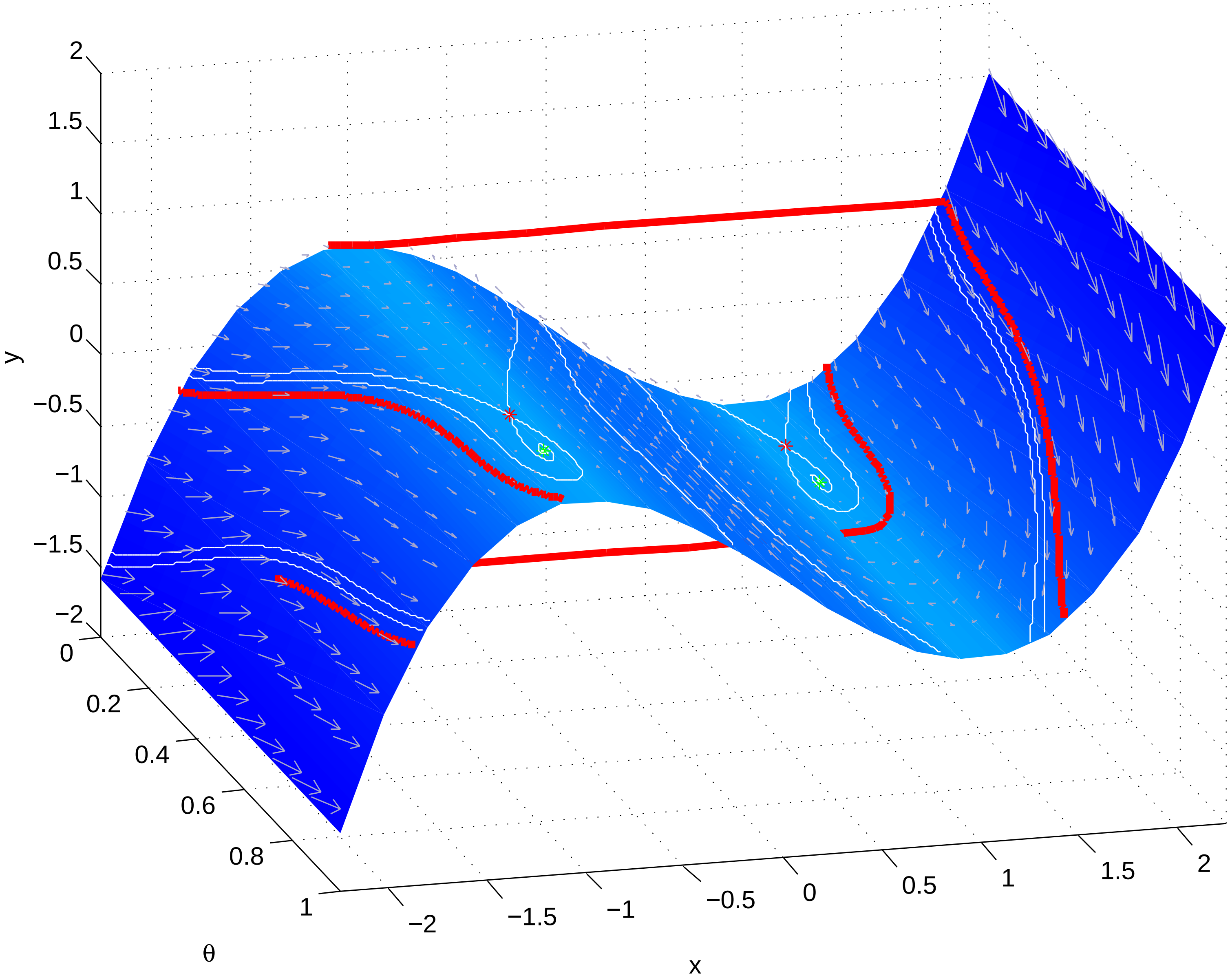

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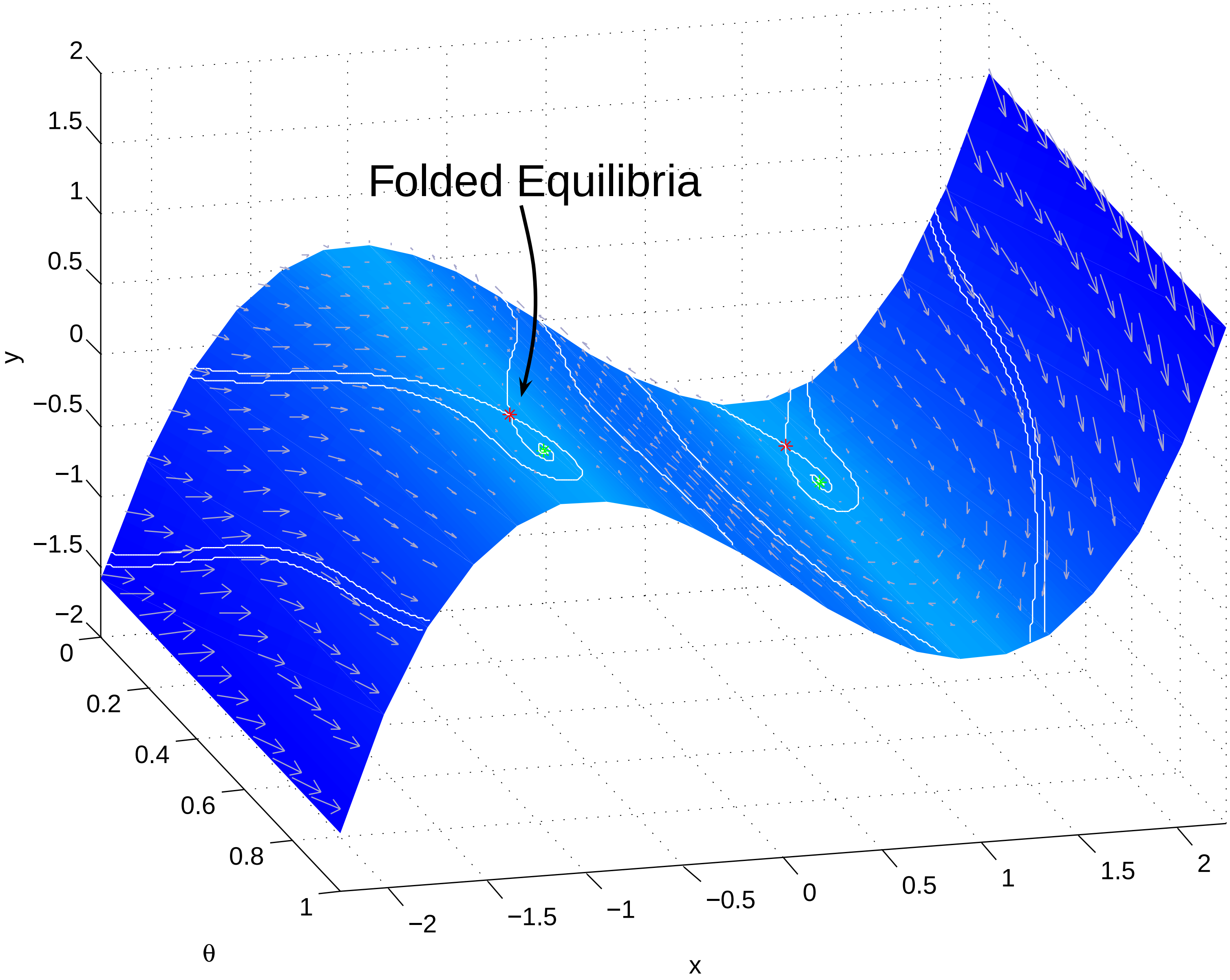

Canards at a Folded Saddle

Representative System (one fast, two slow variables):

$$\begin{aligned} \varepsilon\dot{x} &= y + x^2 \\ \dot{y} &= az + bx \\ \dot{z} &= 1 \end{aligned}$$

Critical manifold is $y = -x^2$. The origin is a folded equilibrium.

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$$a < 0 \implies \text{folded saddle}$$

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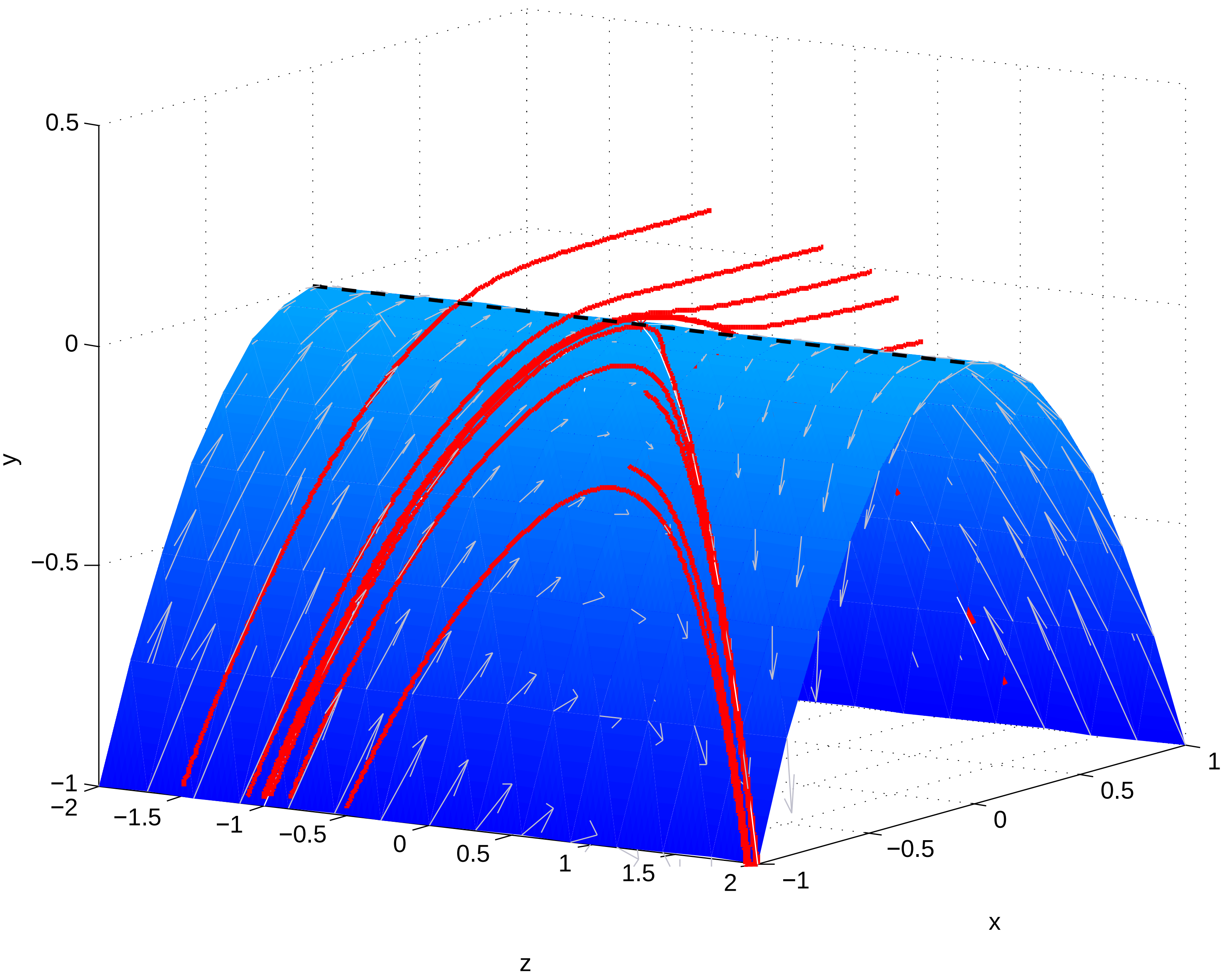

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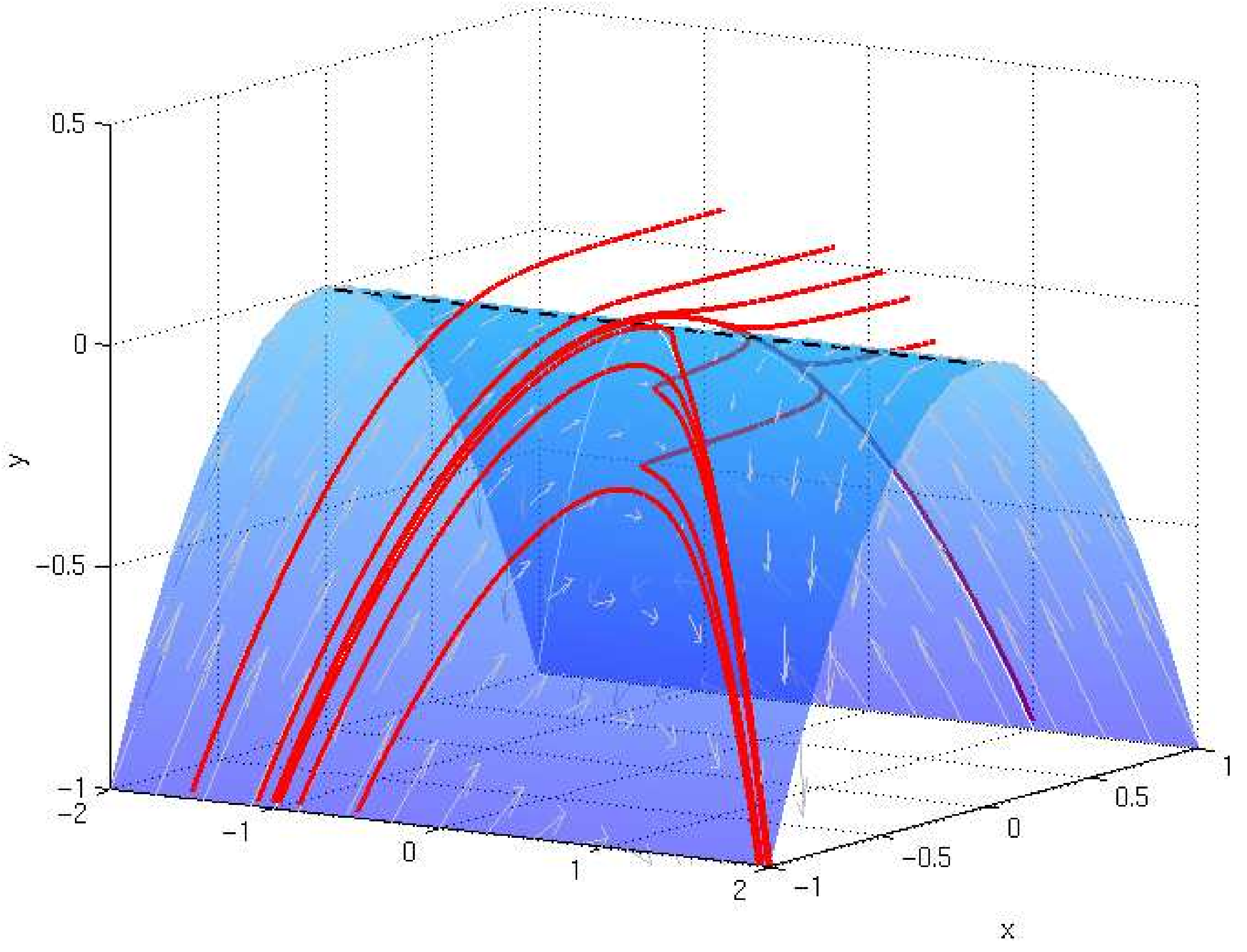

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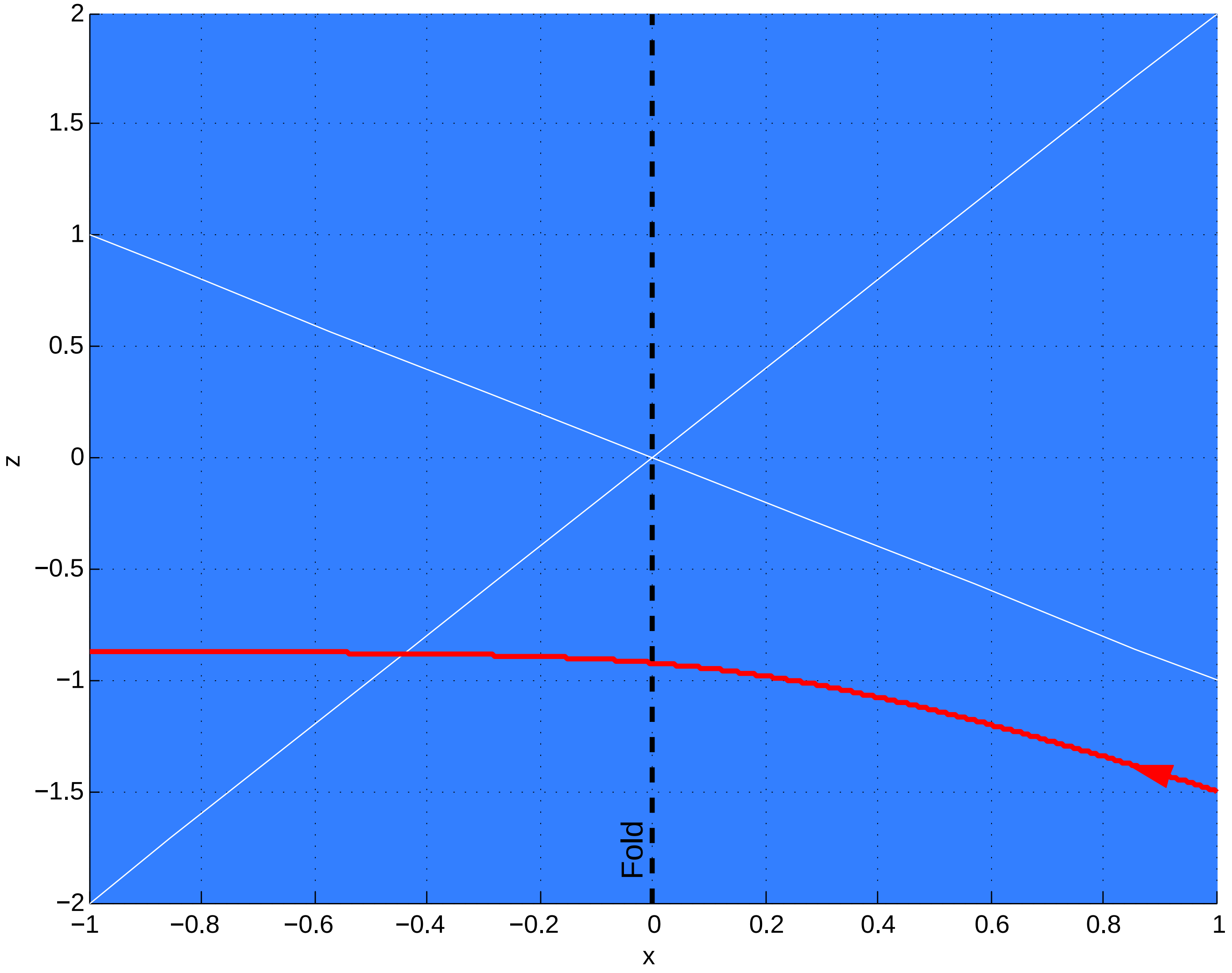

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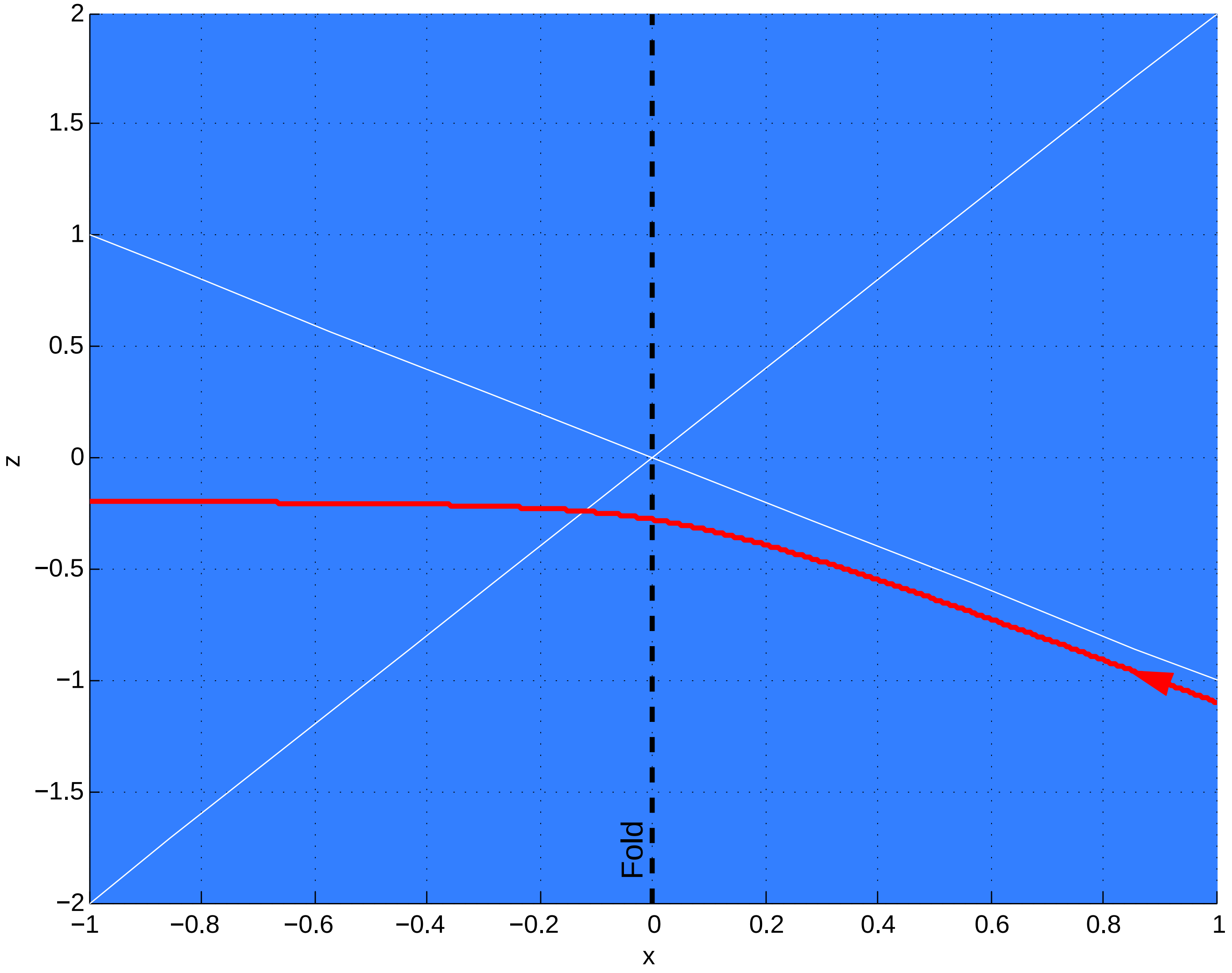

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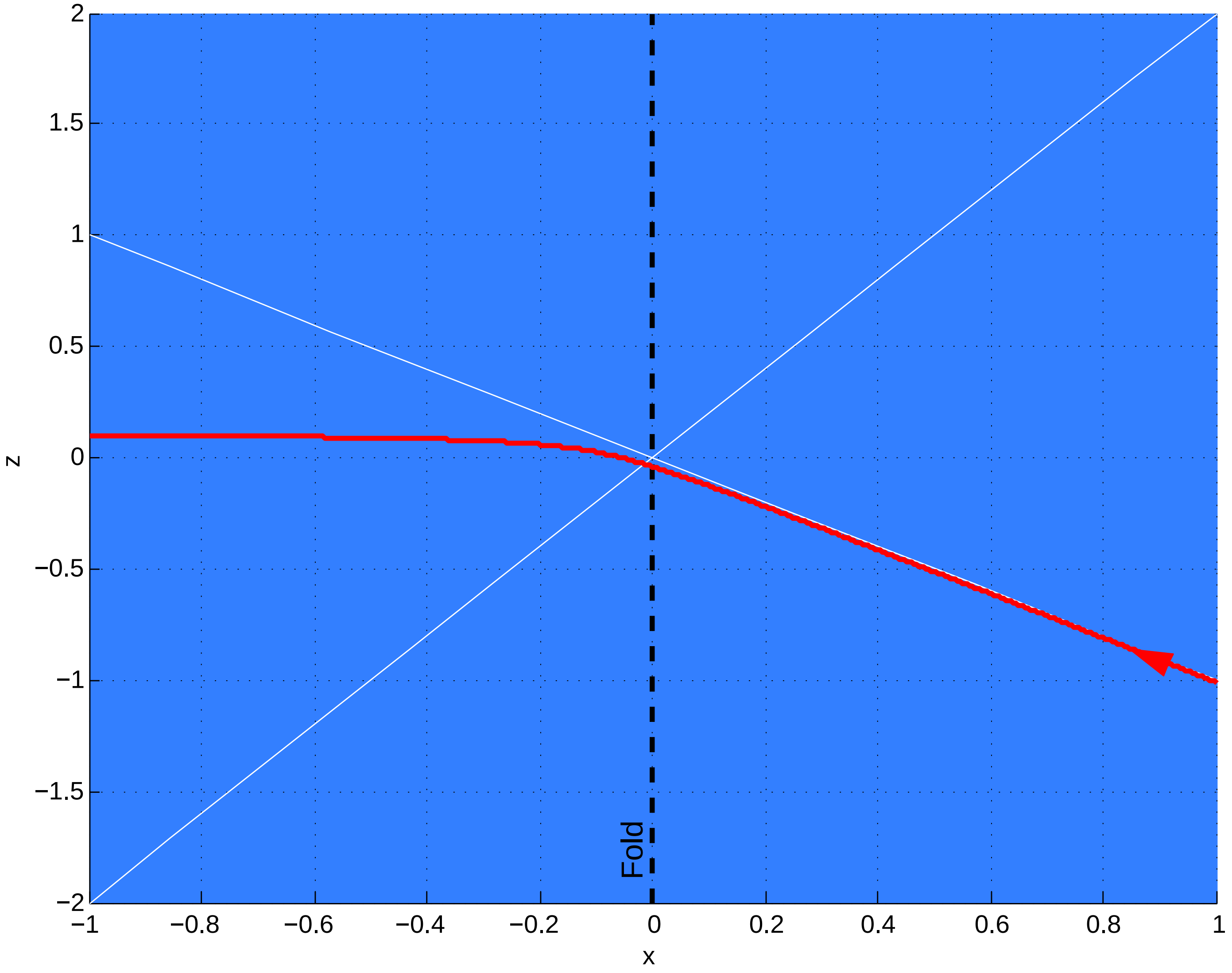

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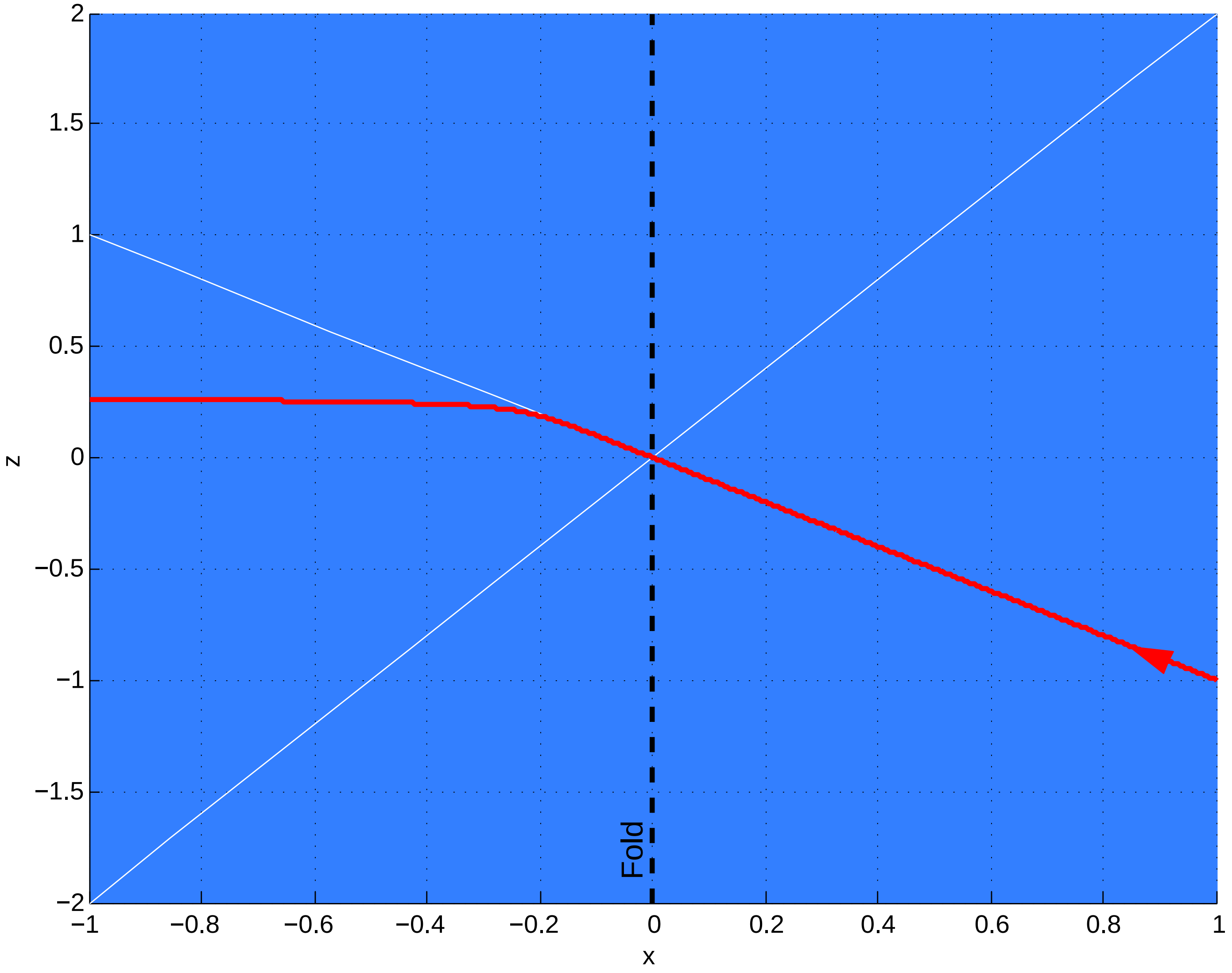

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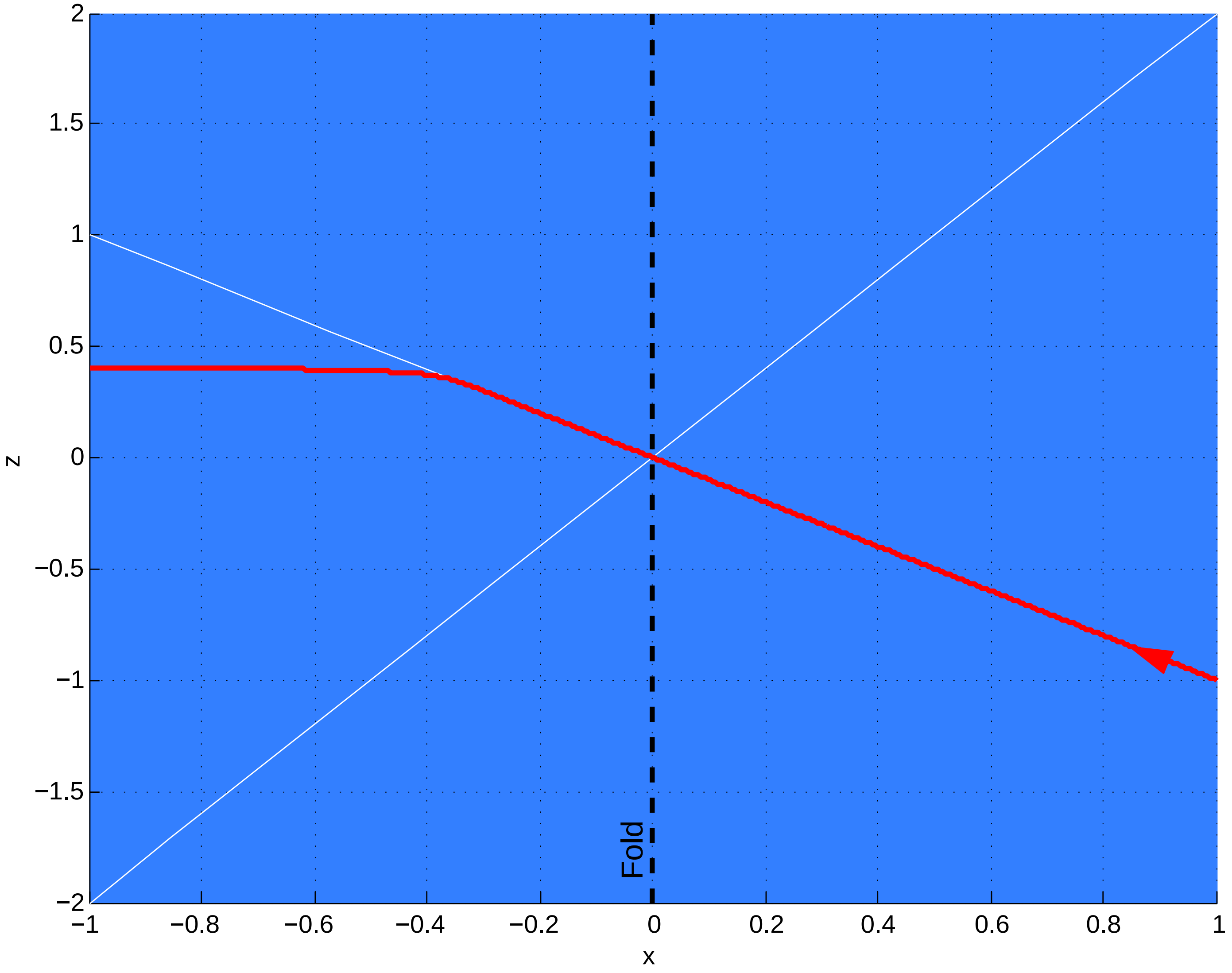

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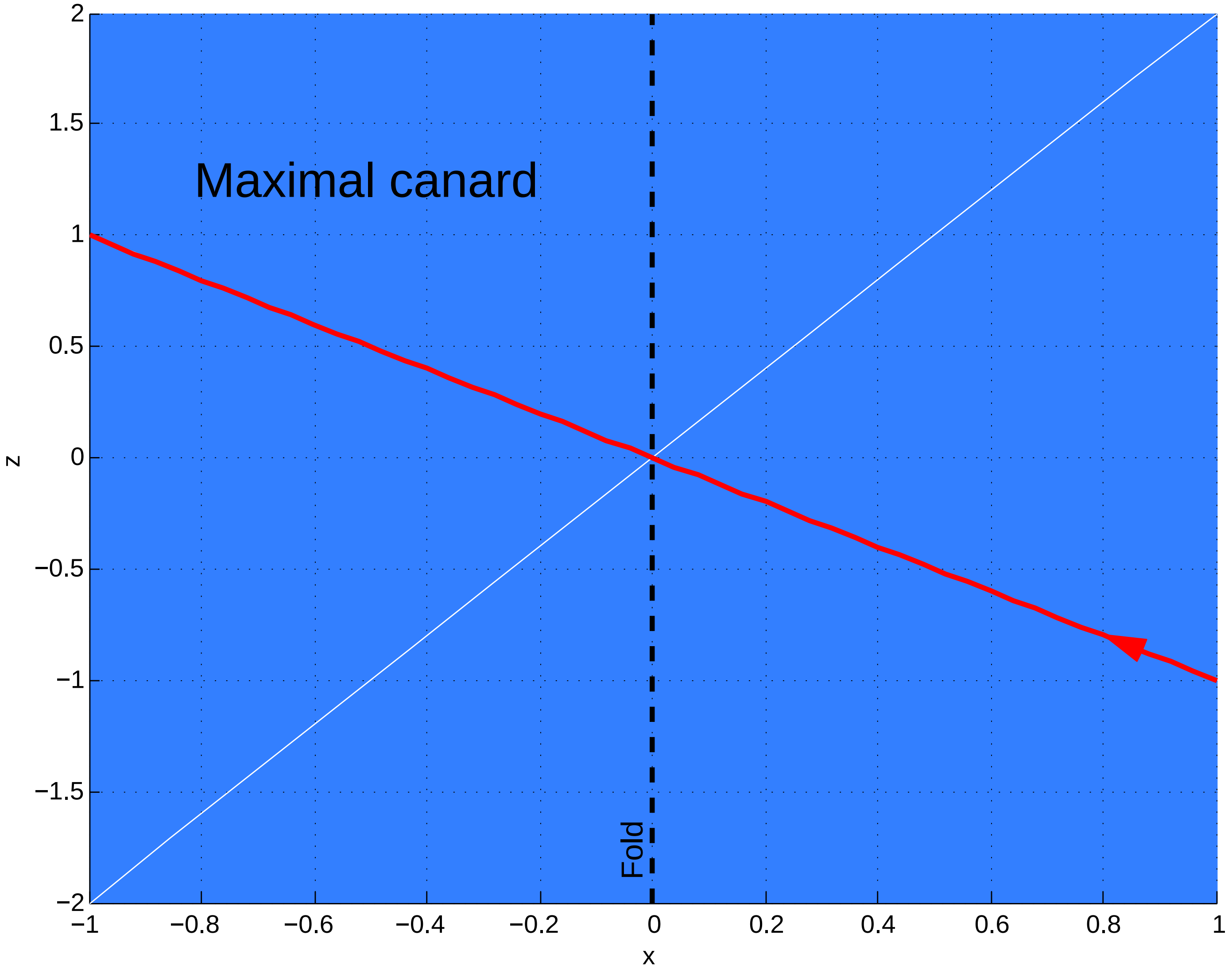

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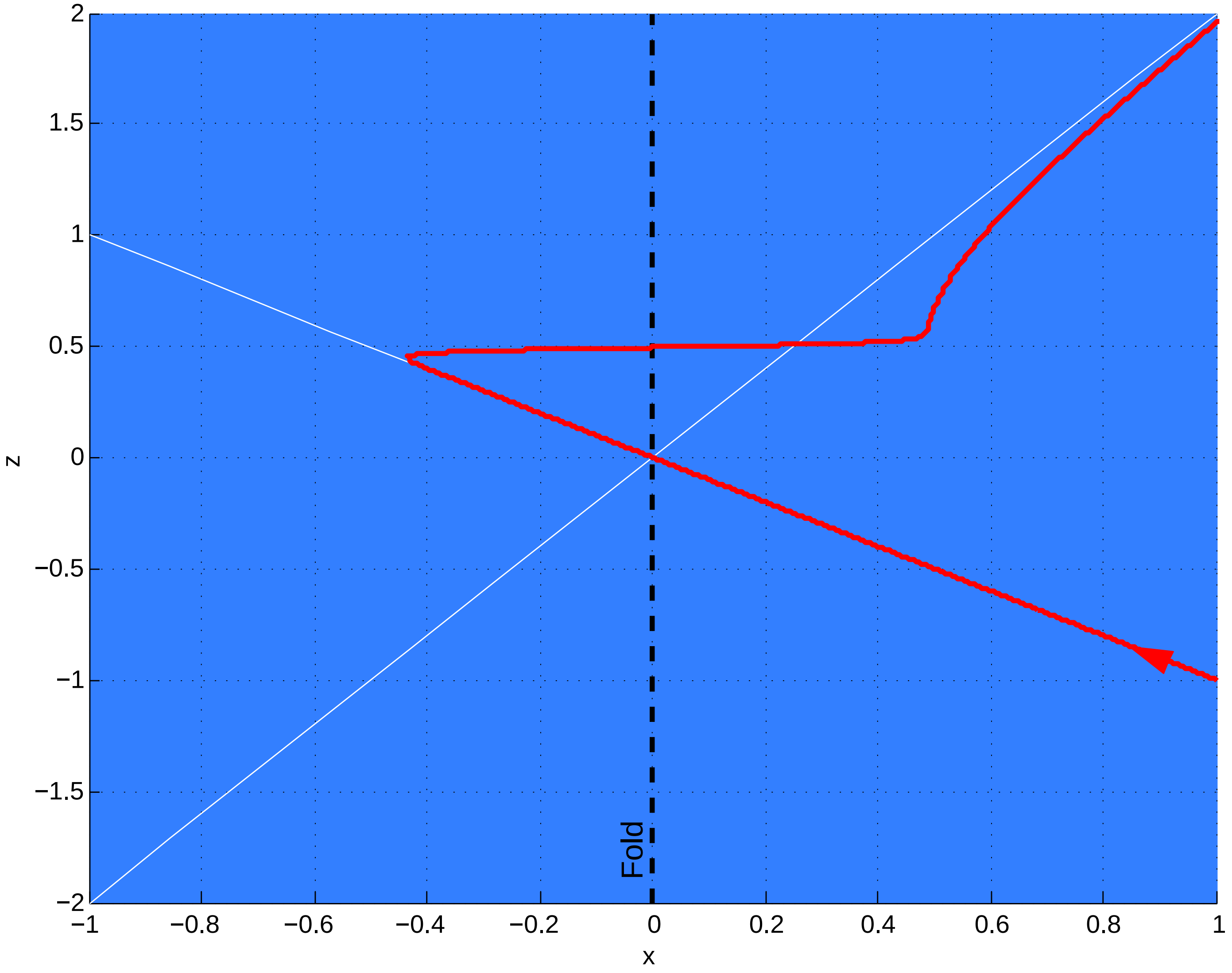

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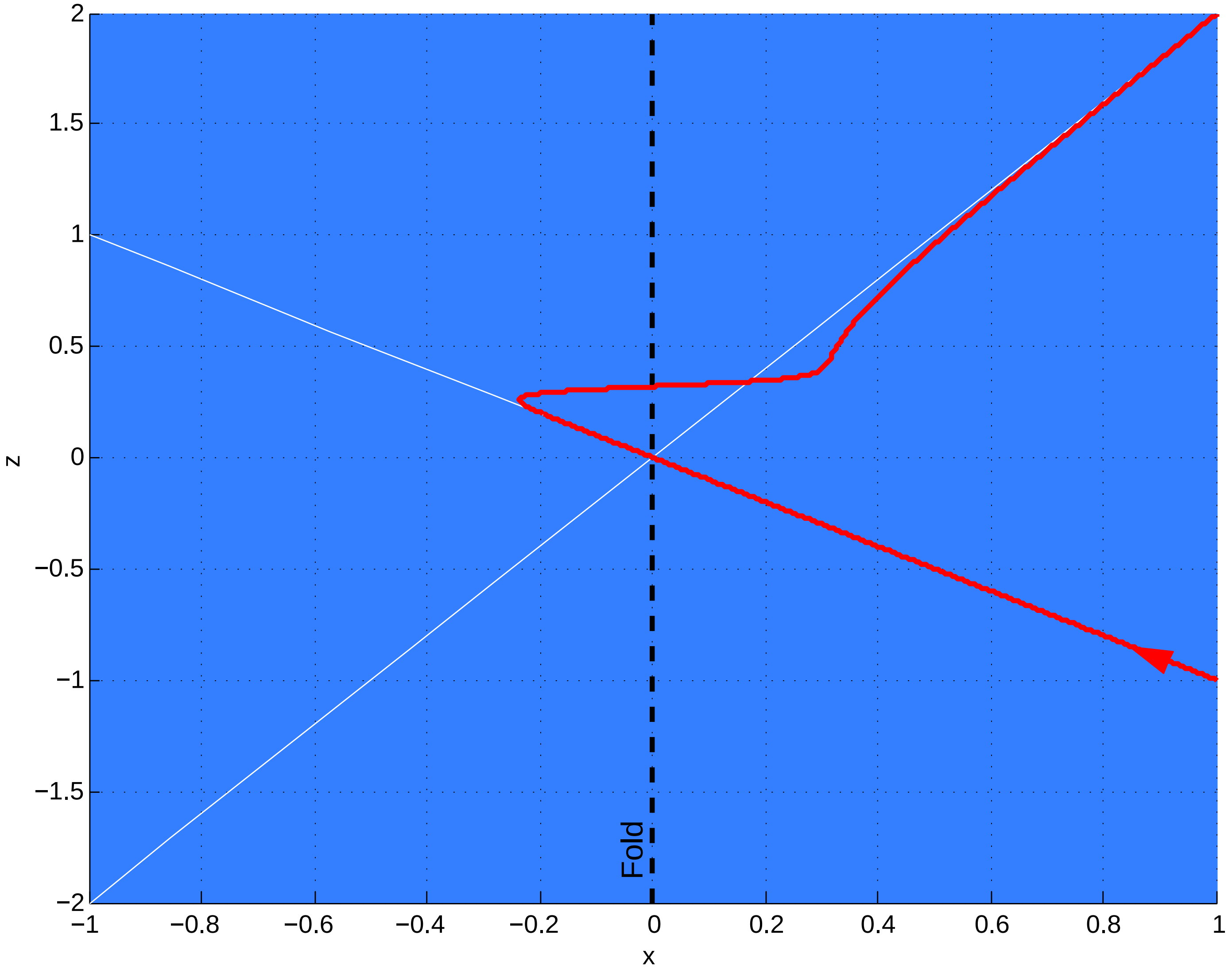

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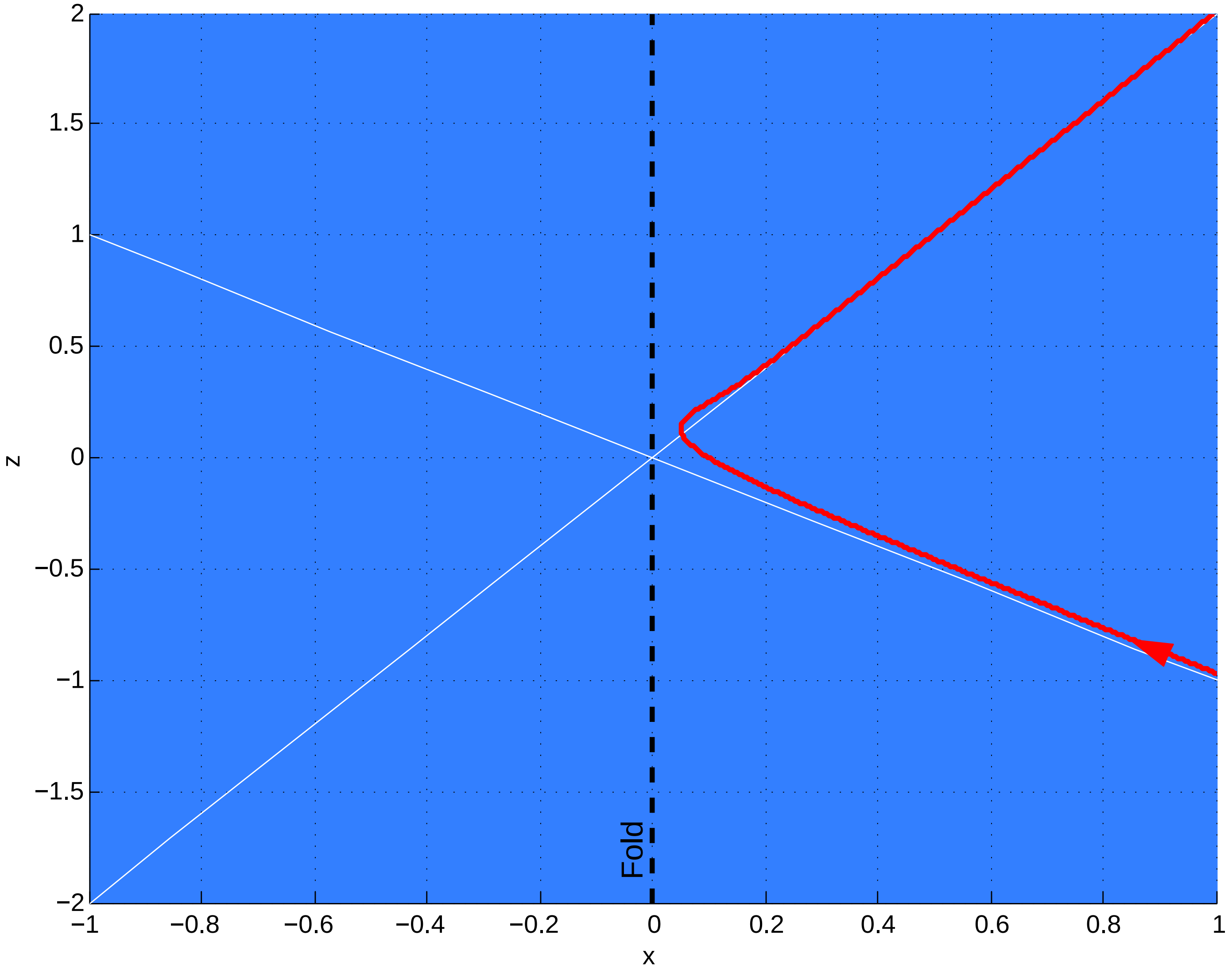

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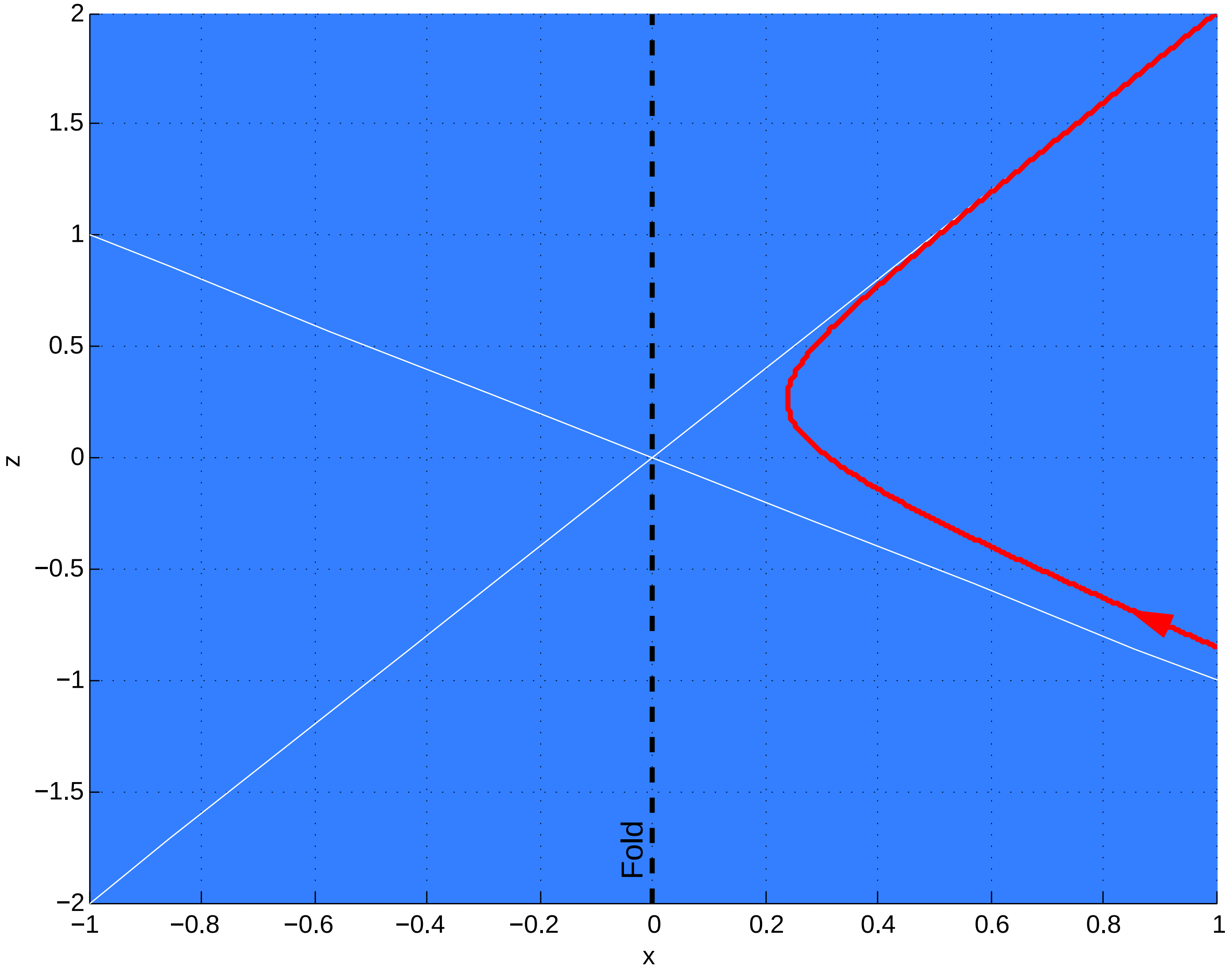

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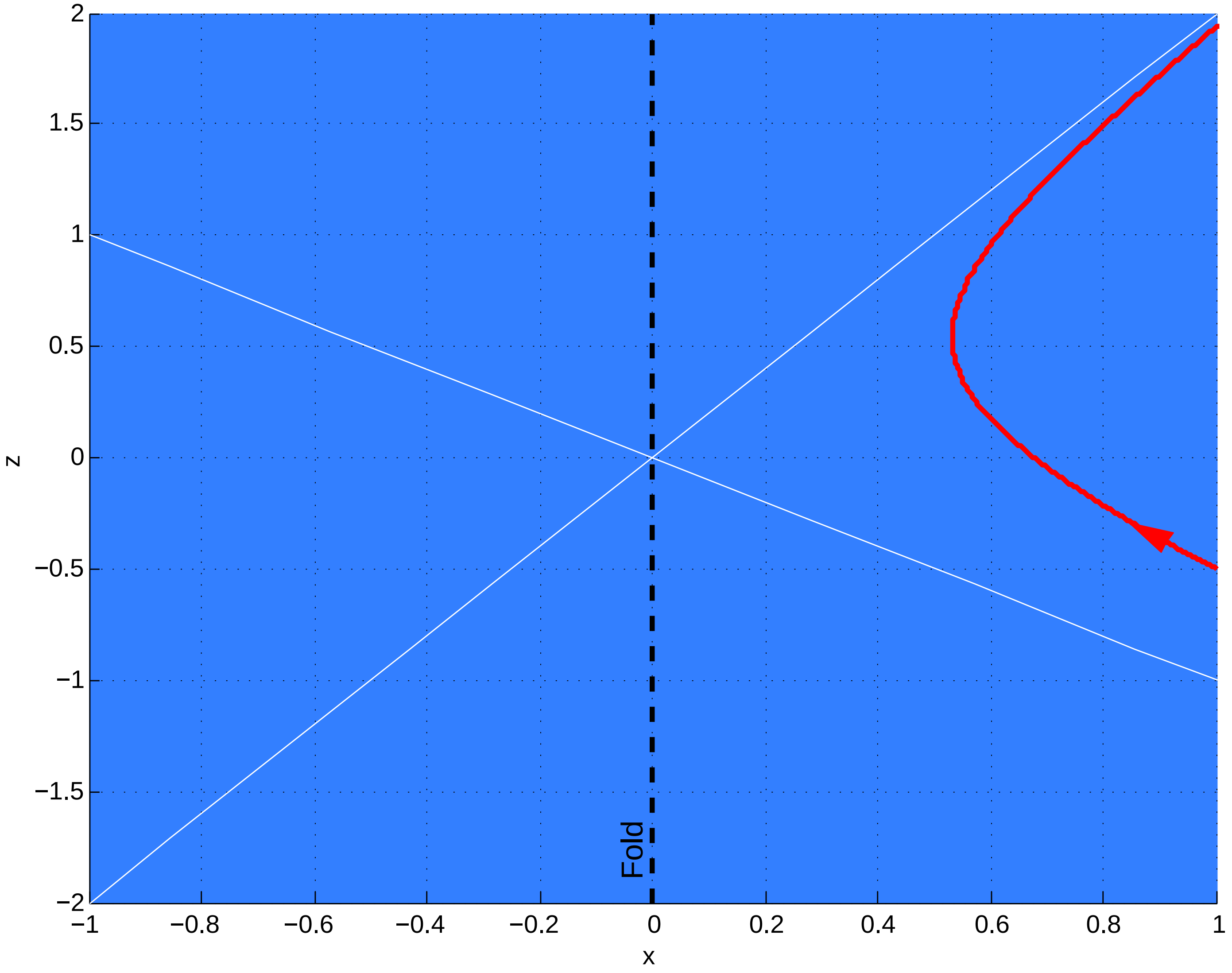
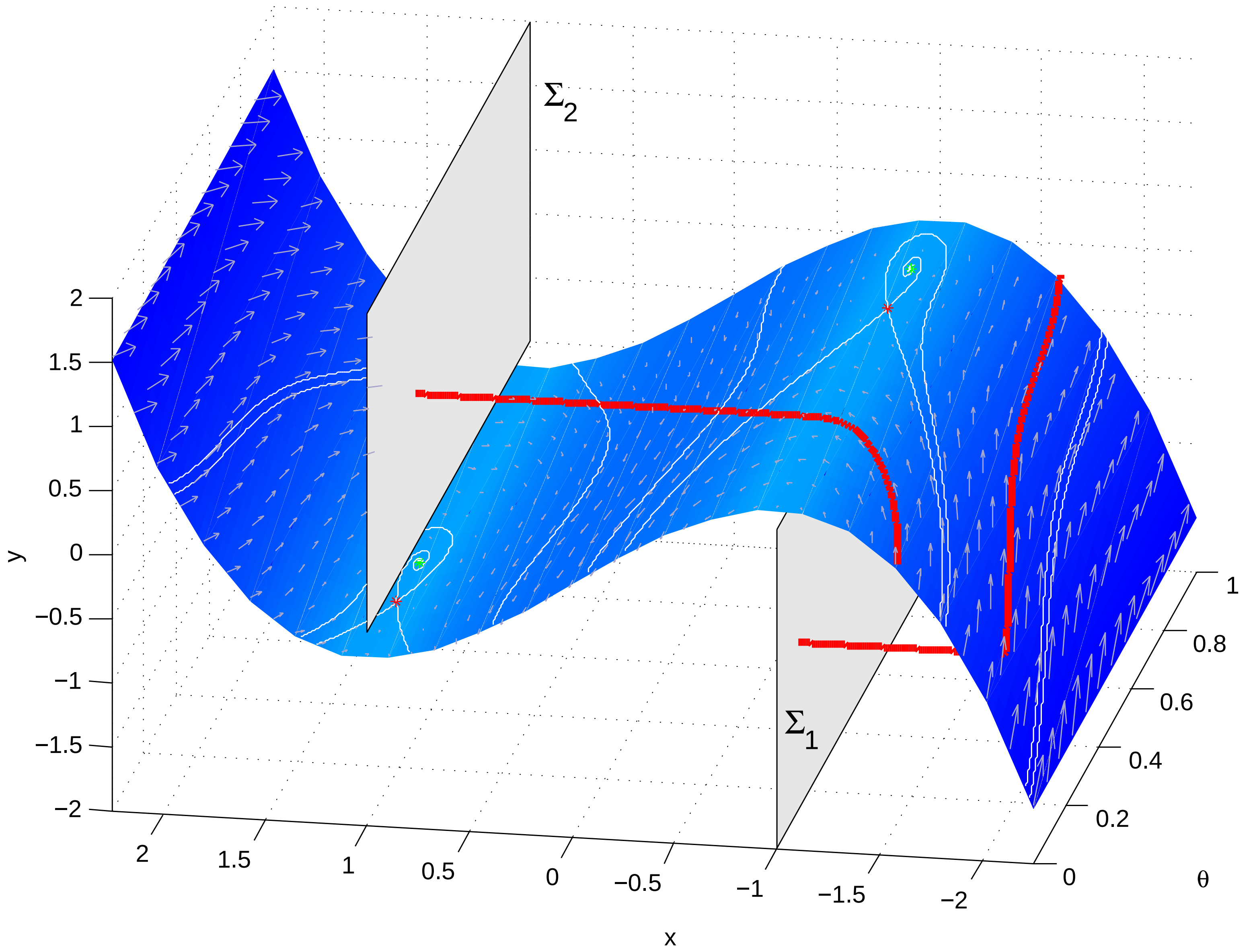


Forced van der Pol Poincaré Map



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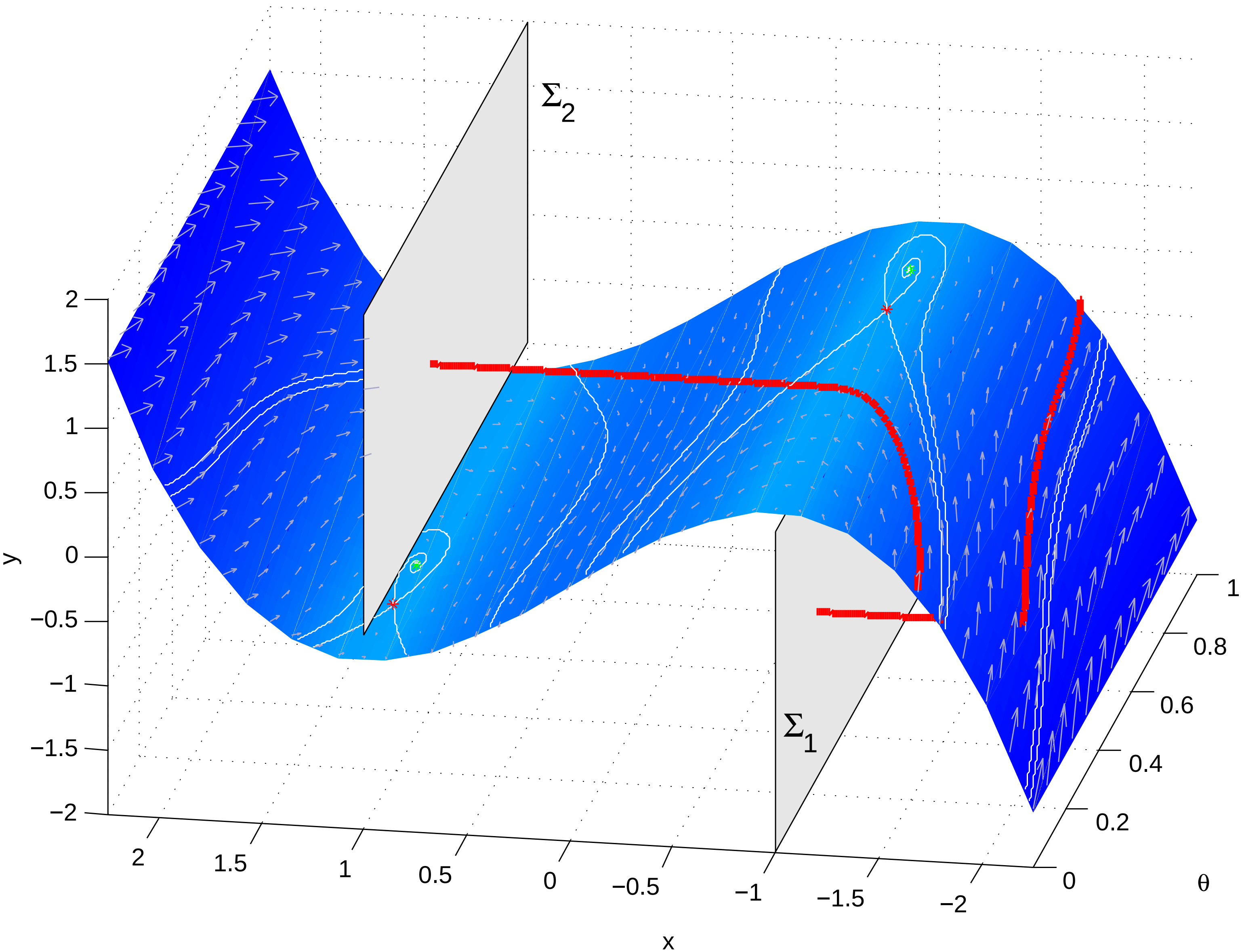

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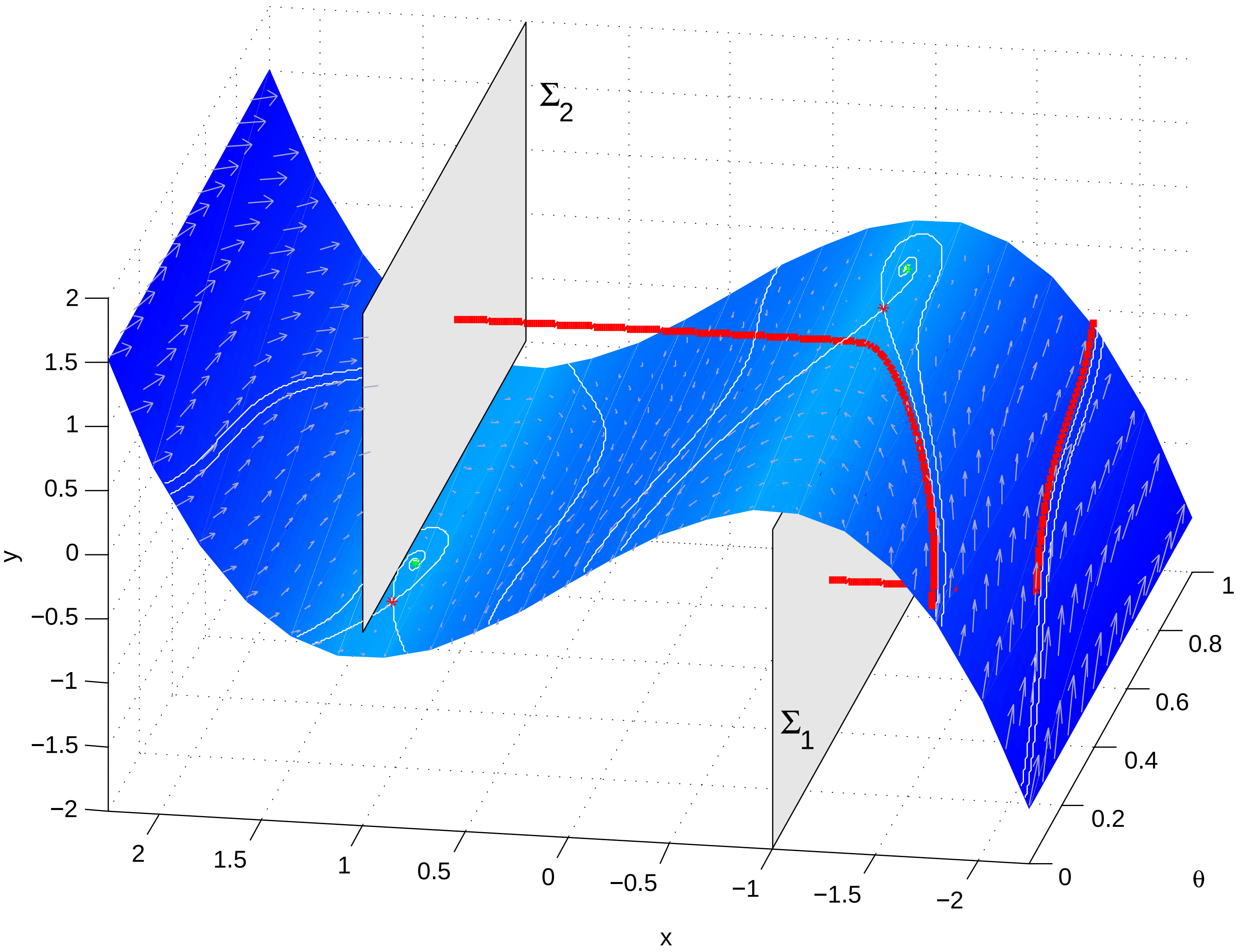

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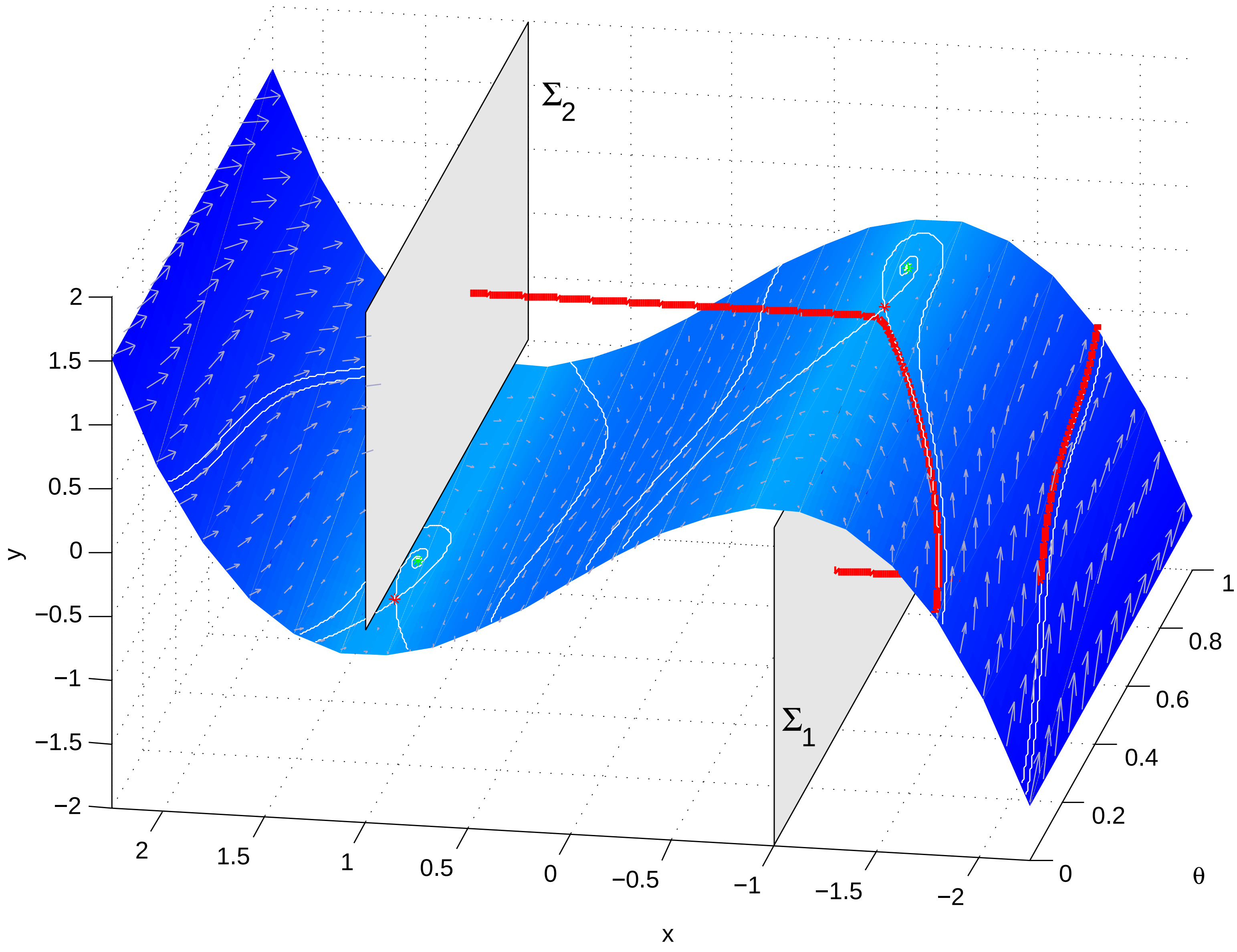

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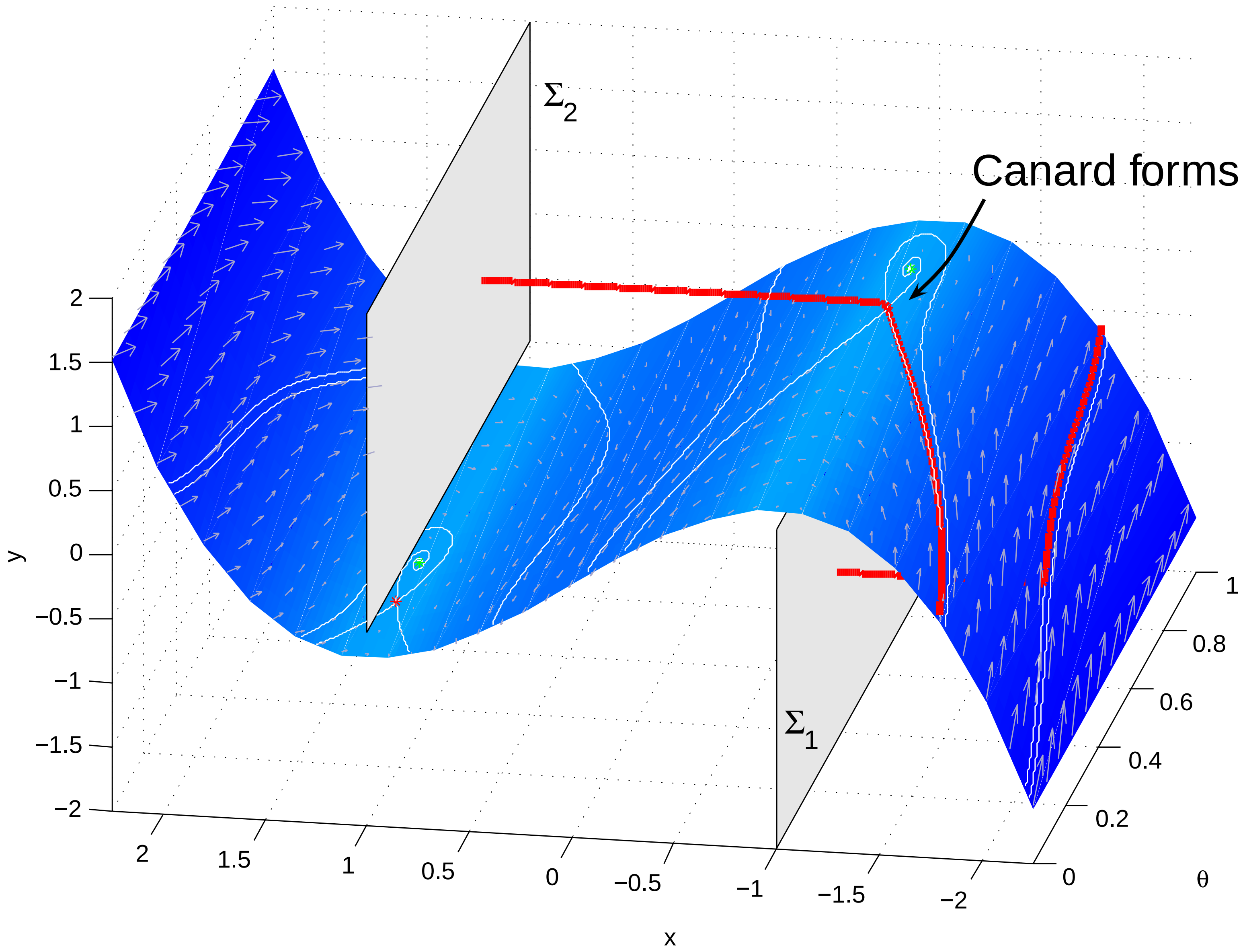

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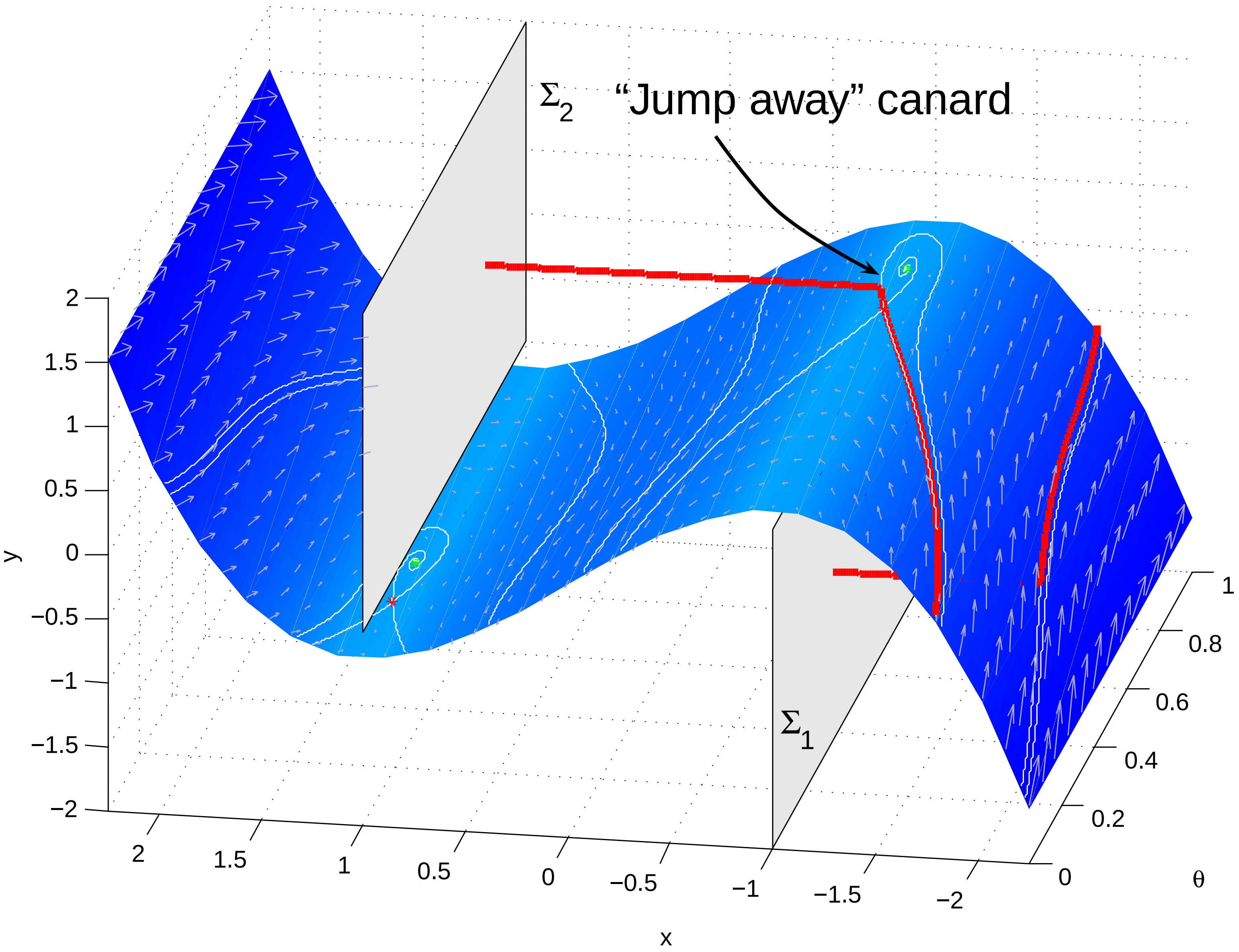

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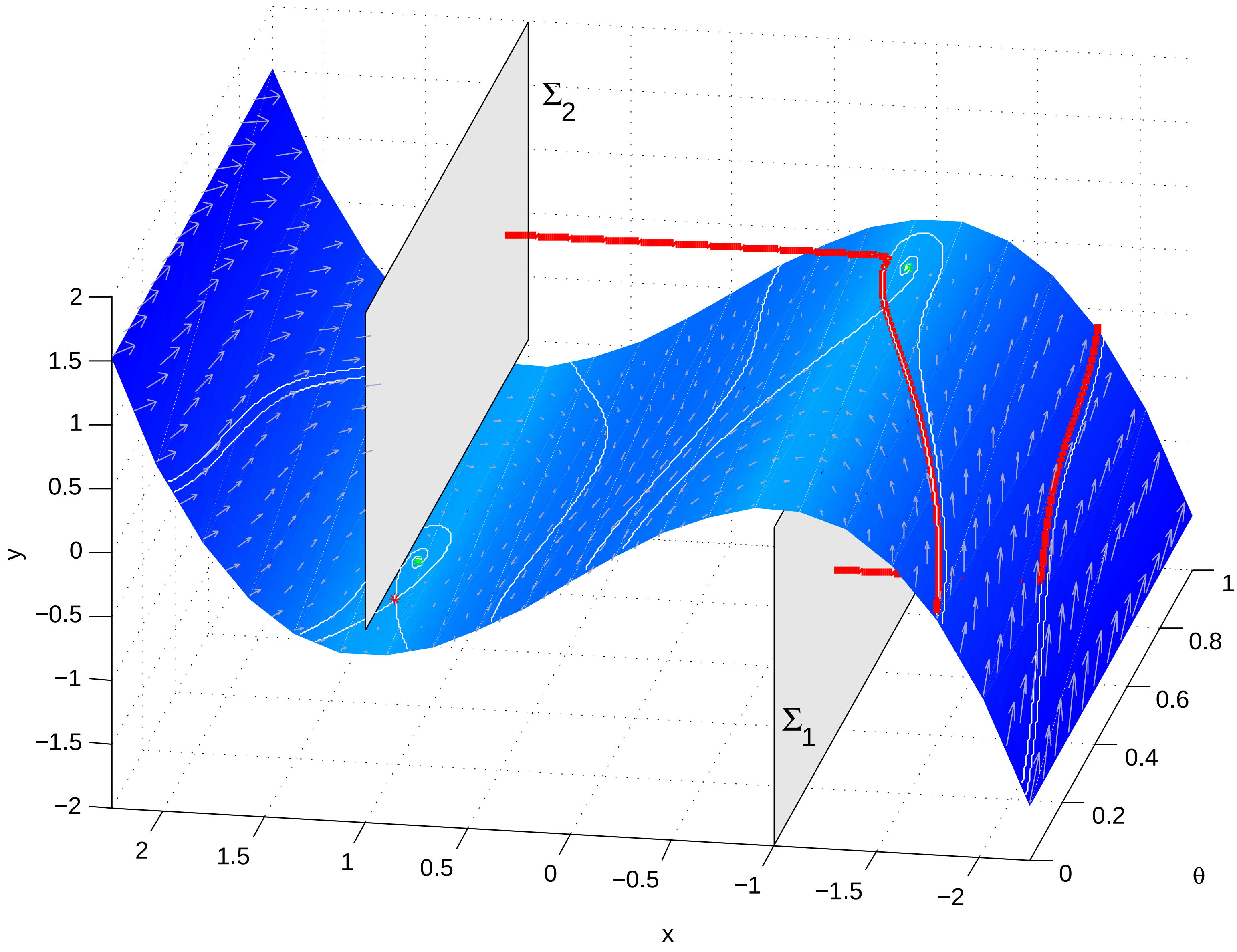

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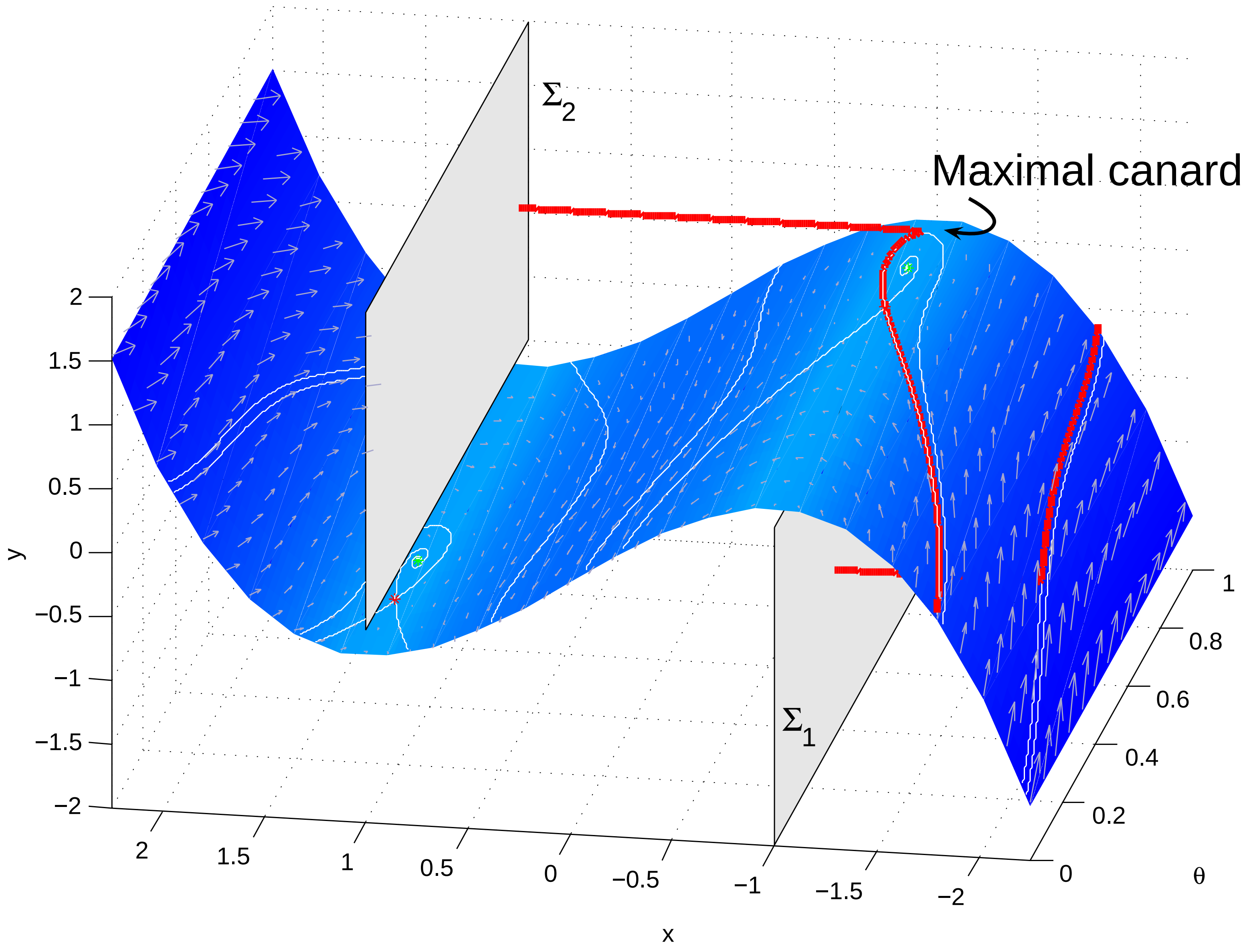

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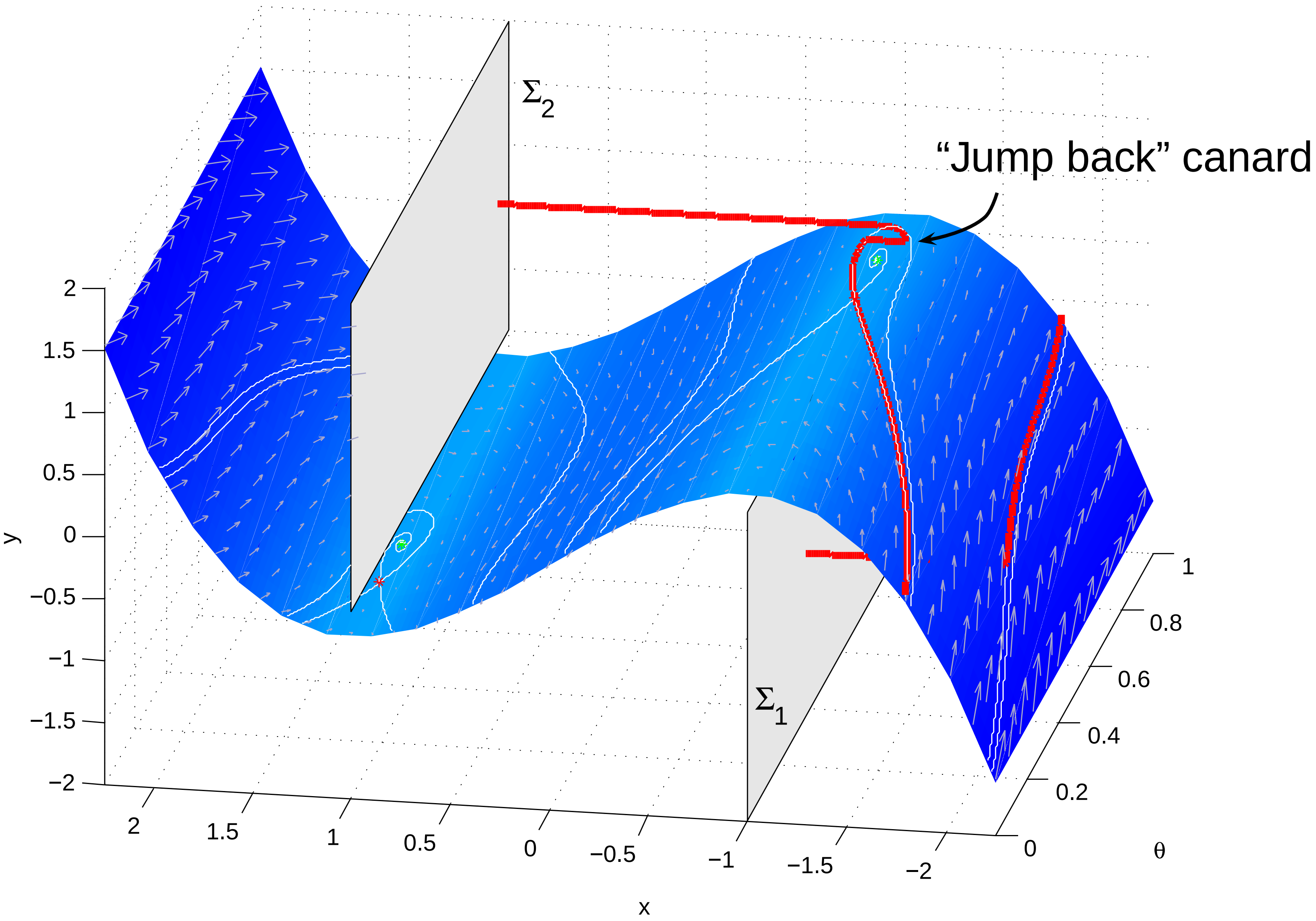

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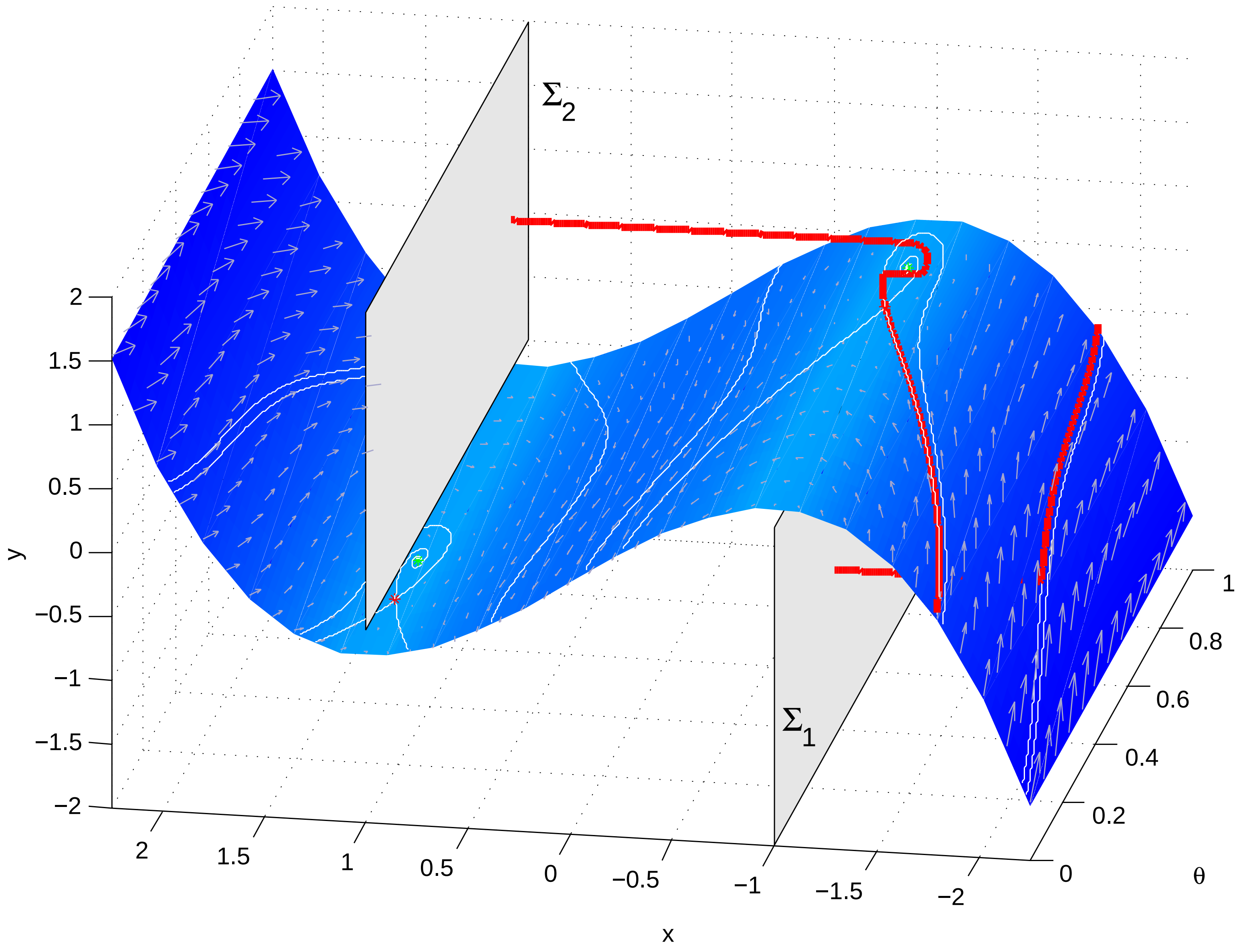

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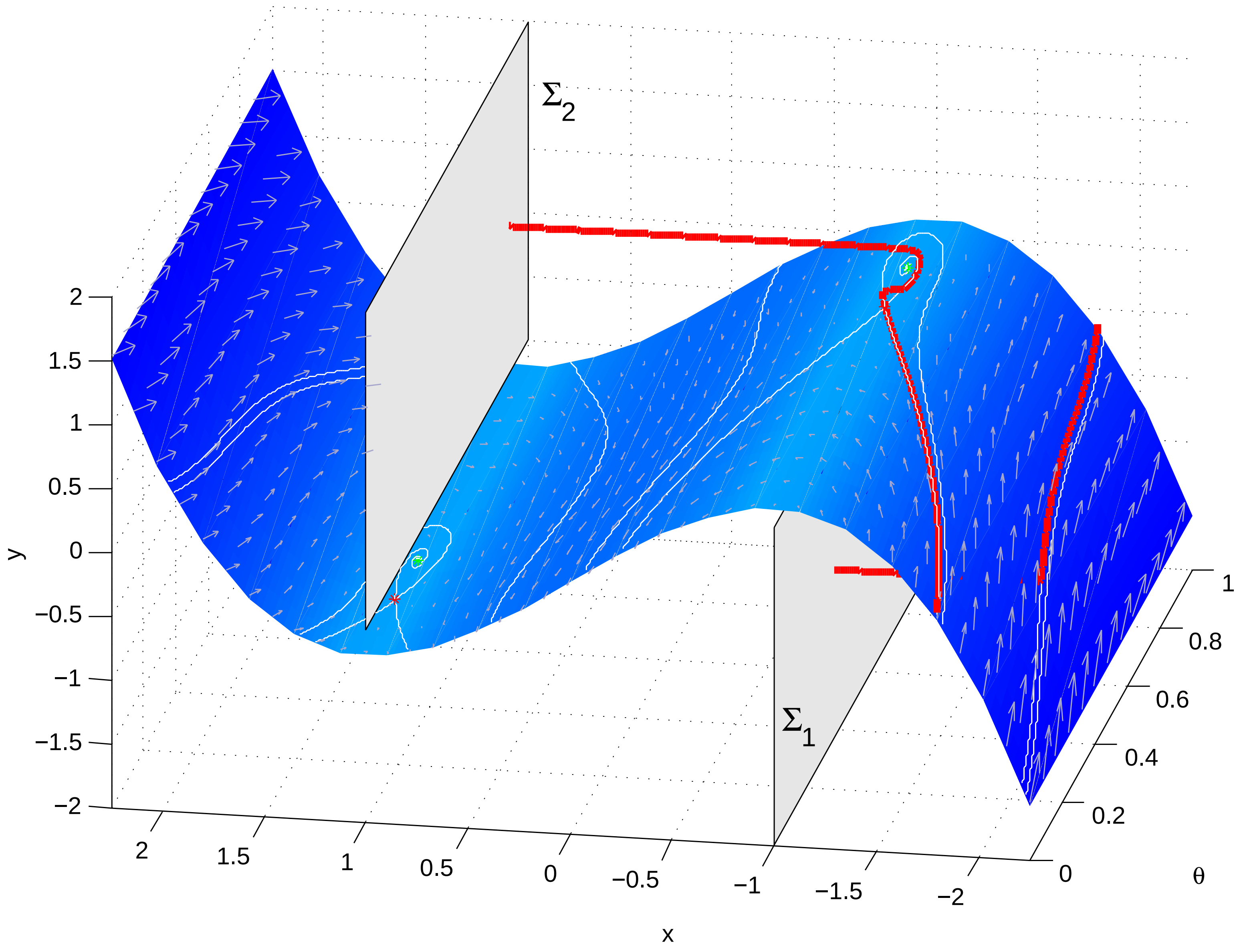

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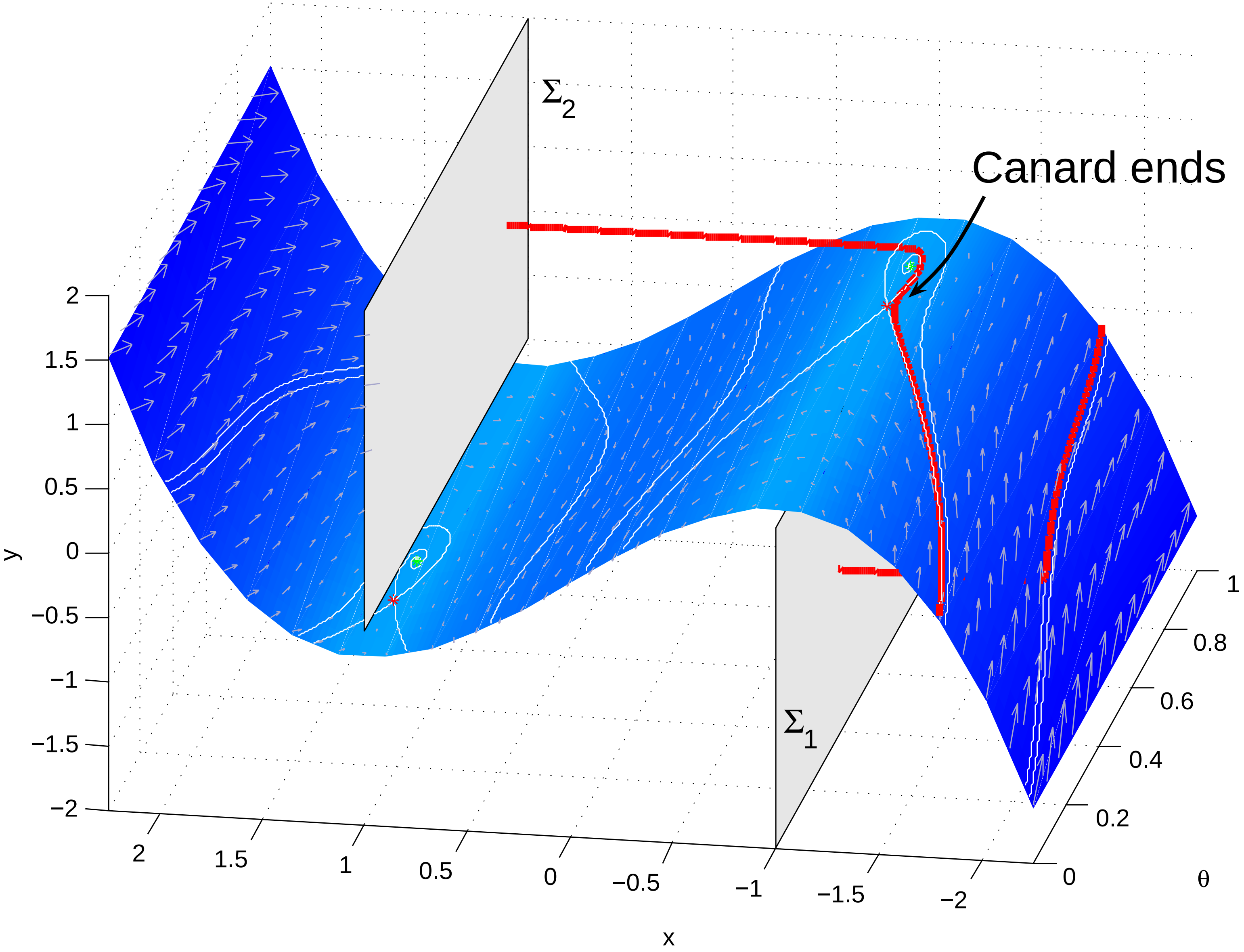

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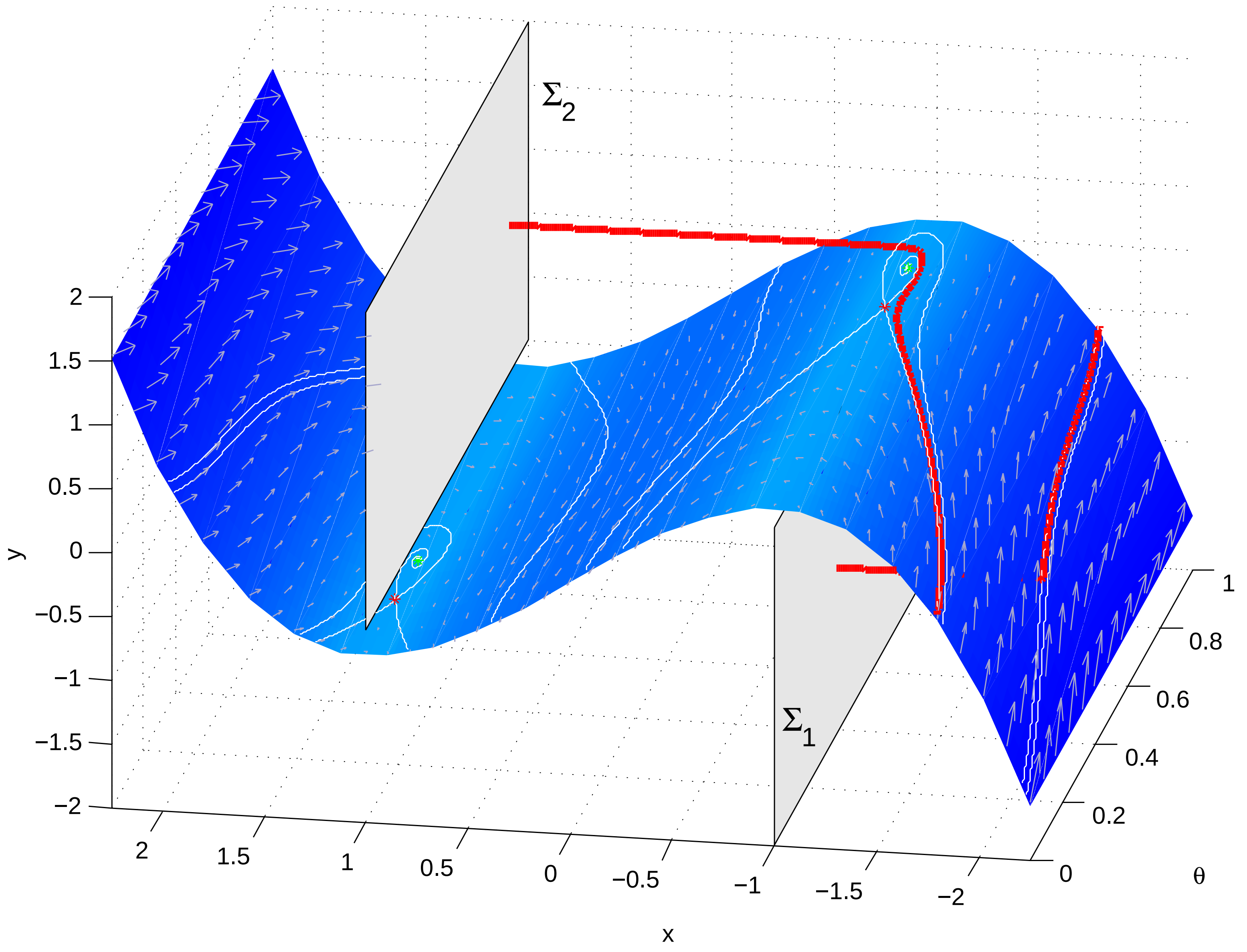

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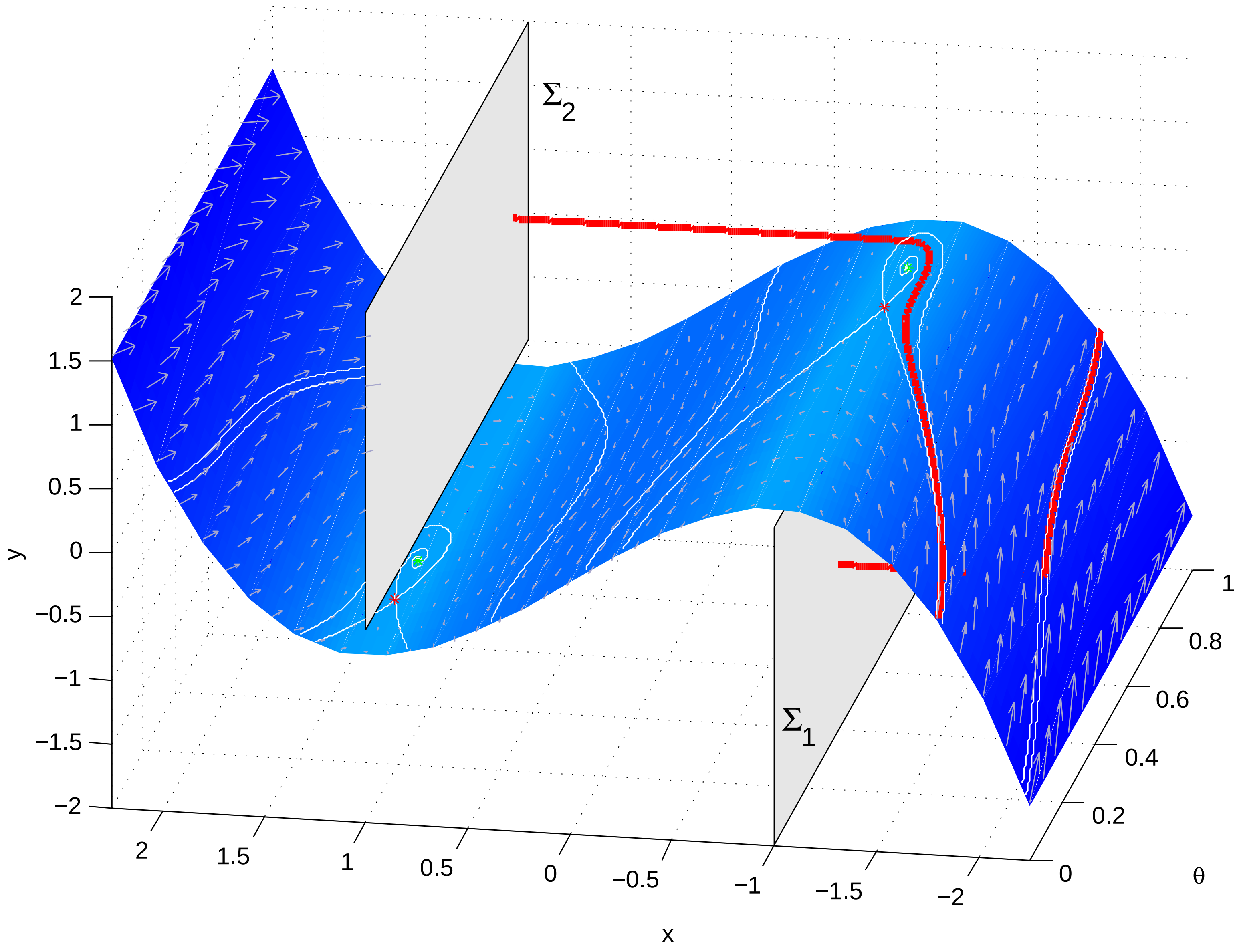

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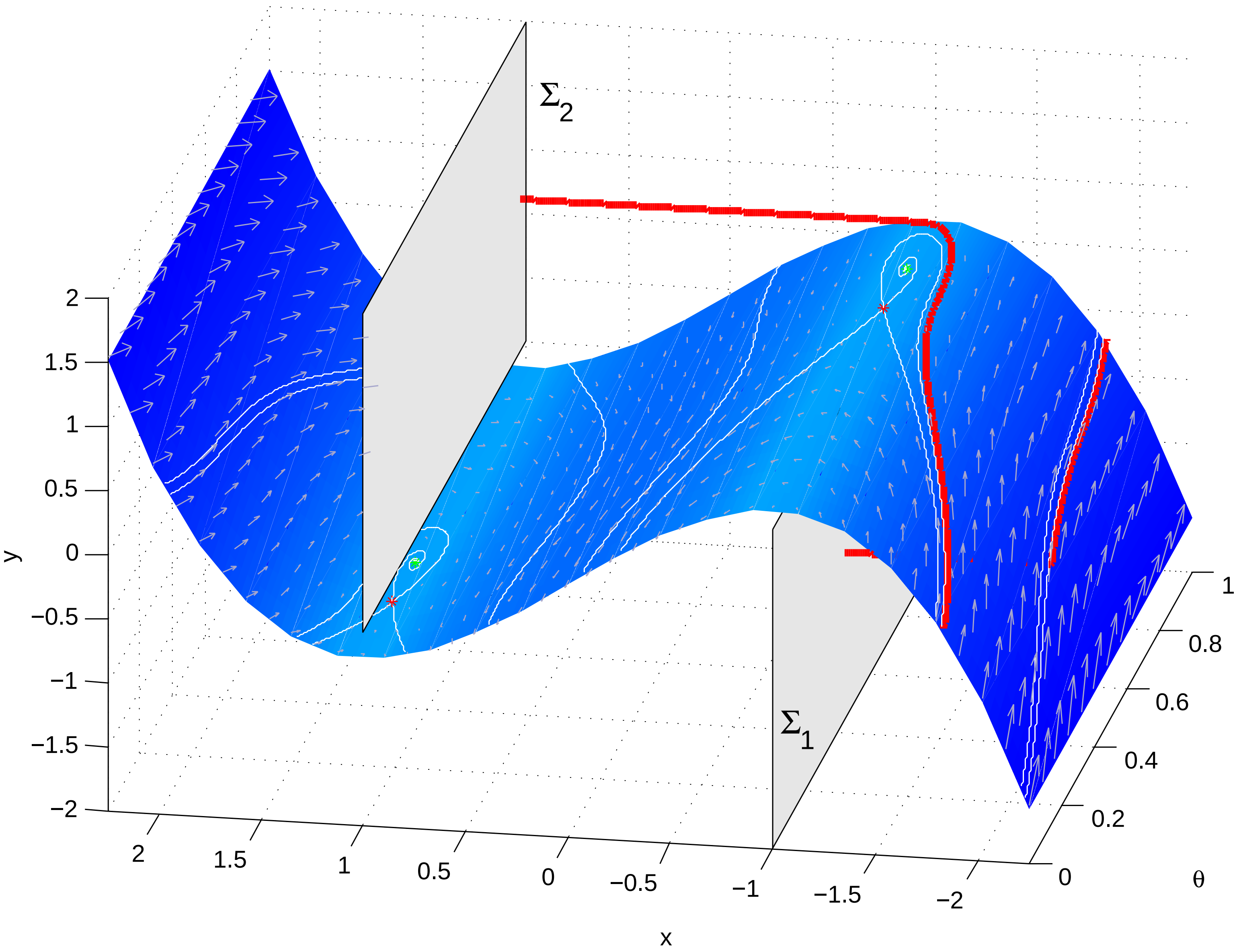

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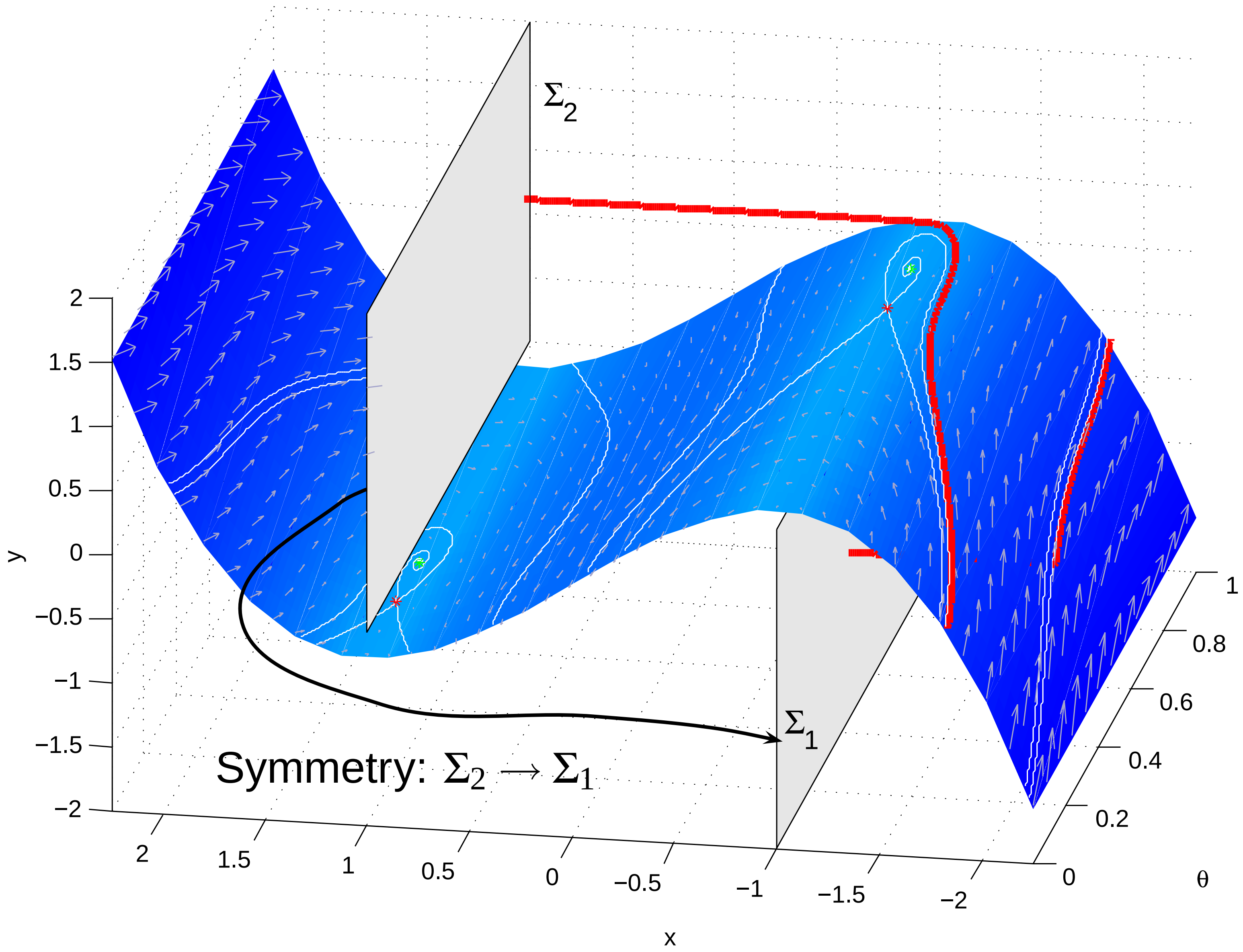
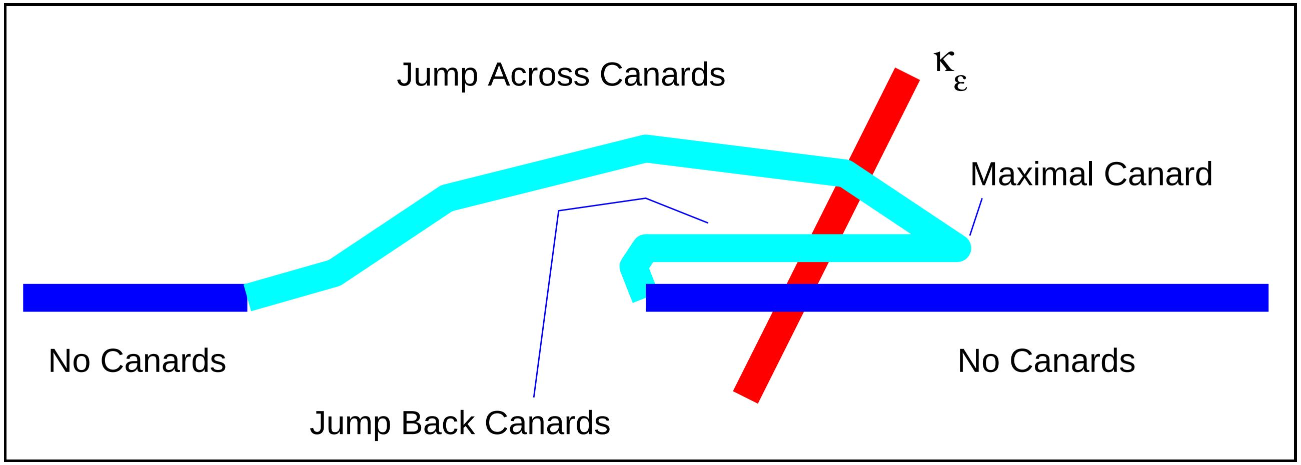


Horseshoe in the Forced van der Pol System

Sketch



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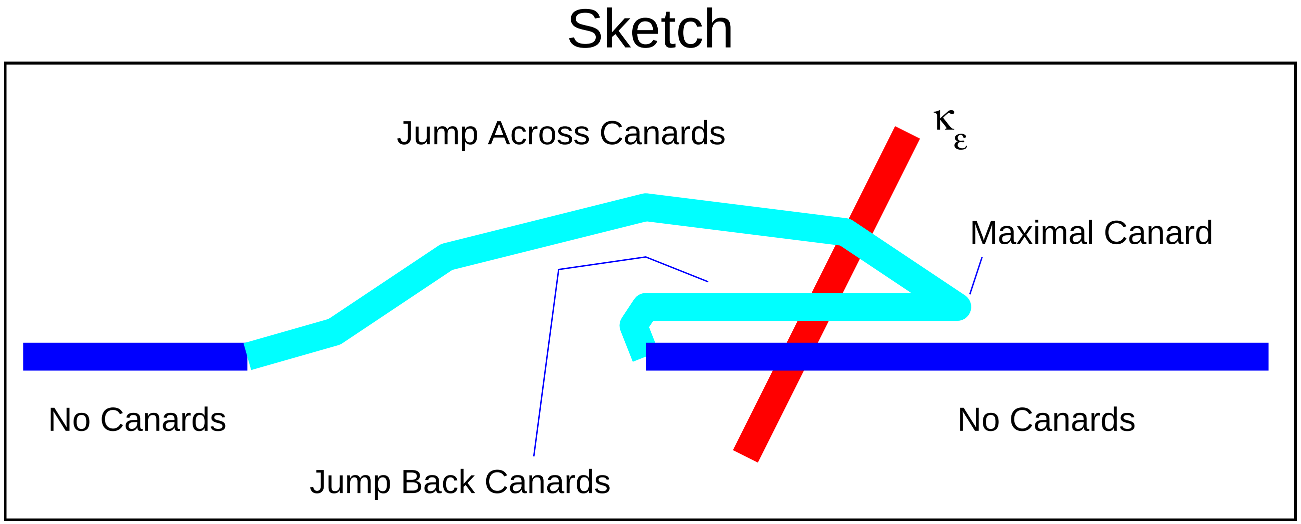
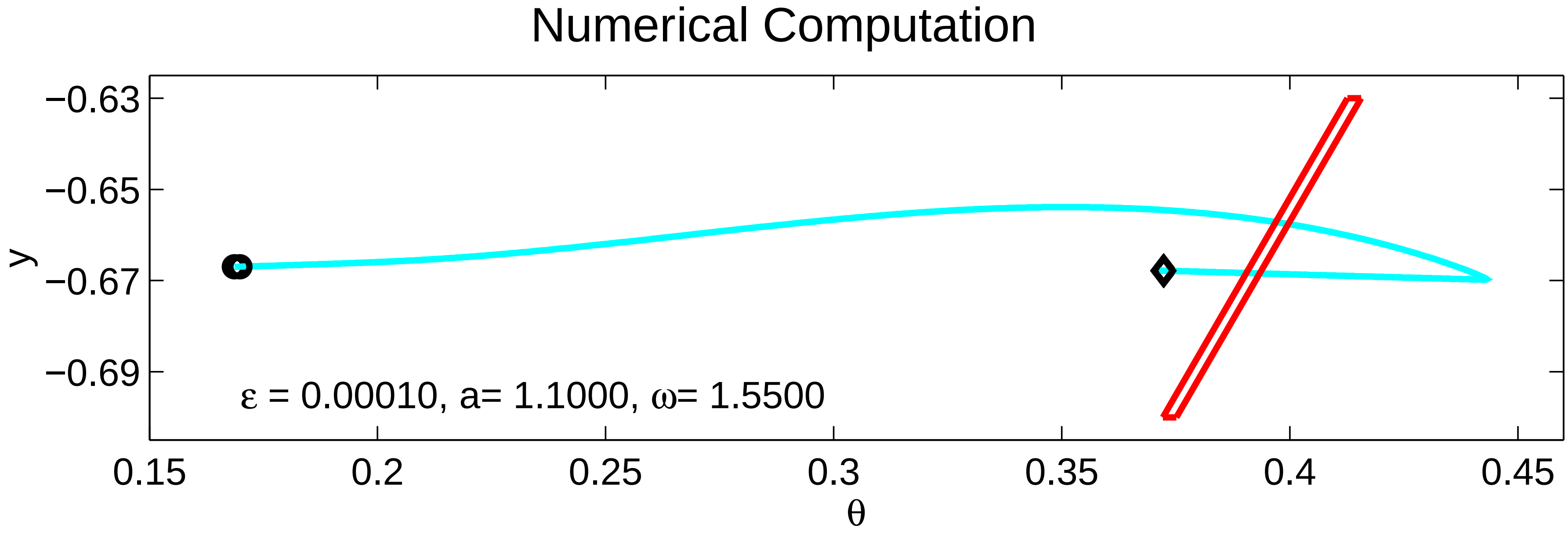
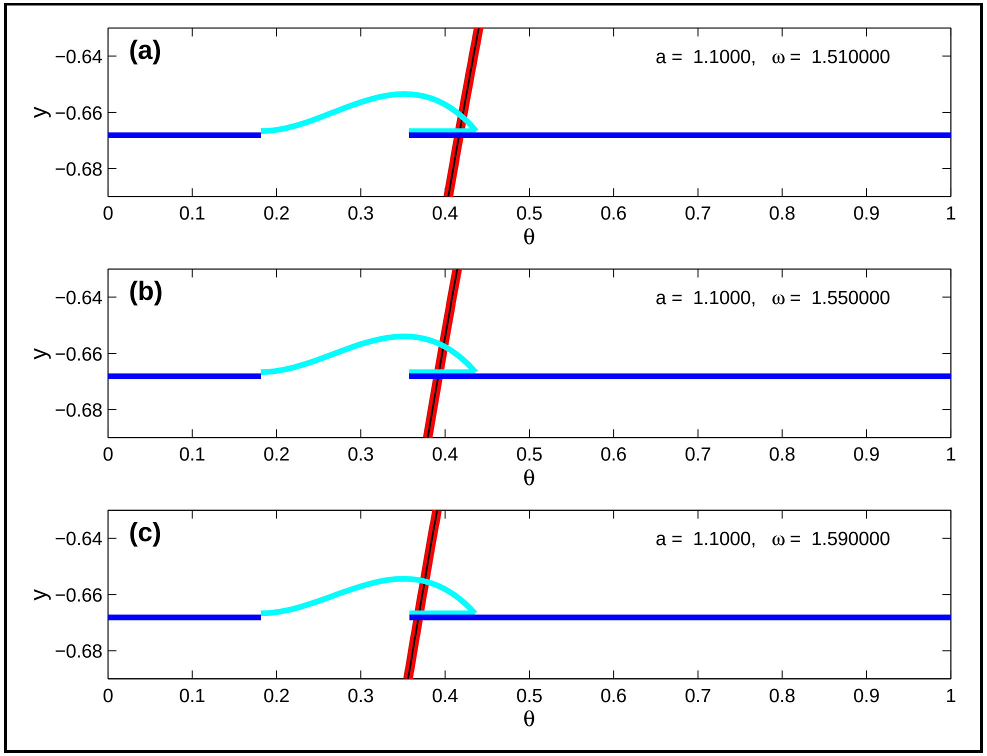




Singular Poincaré Maps



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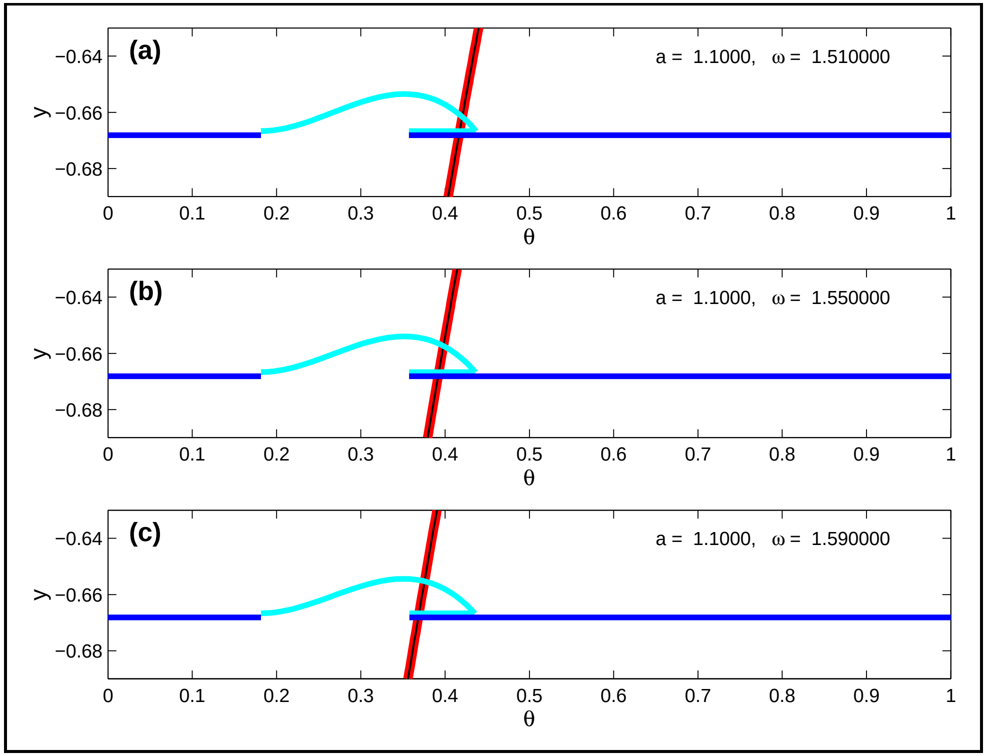
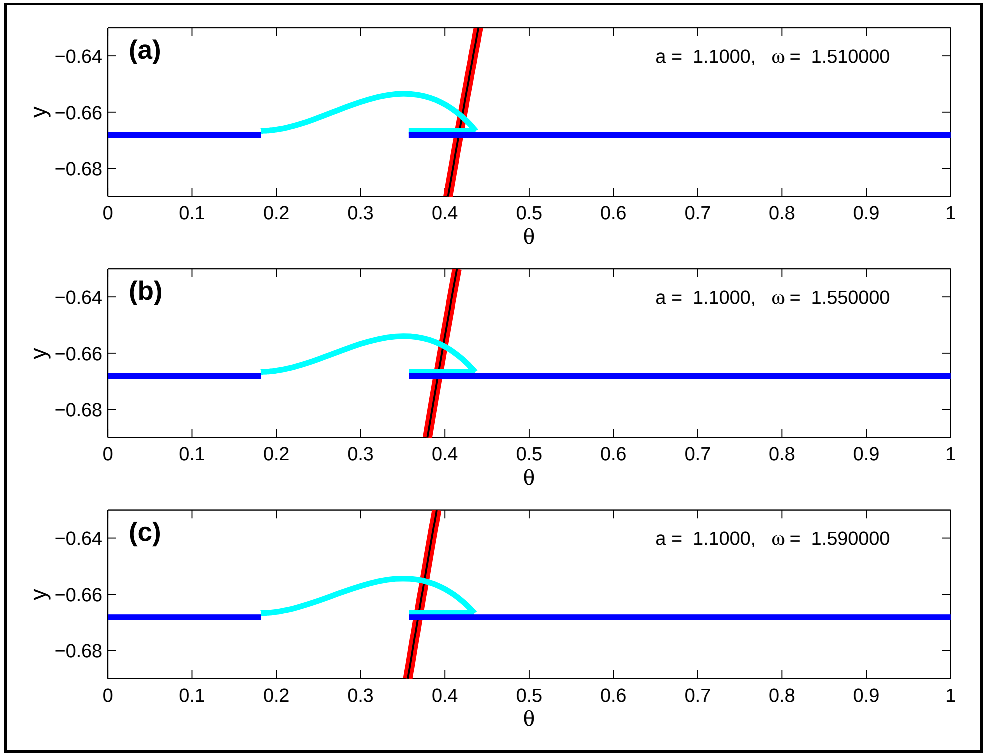
Singular Poincaré Maps



- ❑ Based on the fast and slow subsystems only.
- ❑ Singular limit of canard trajectories results in nonuniqueness.

Singular Poincaré Maps



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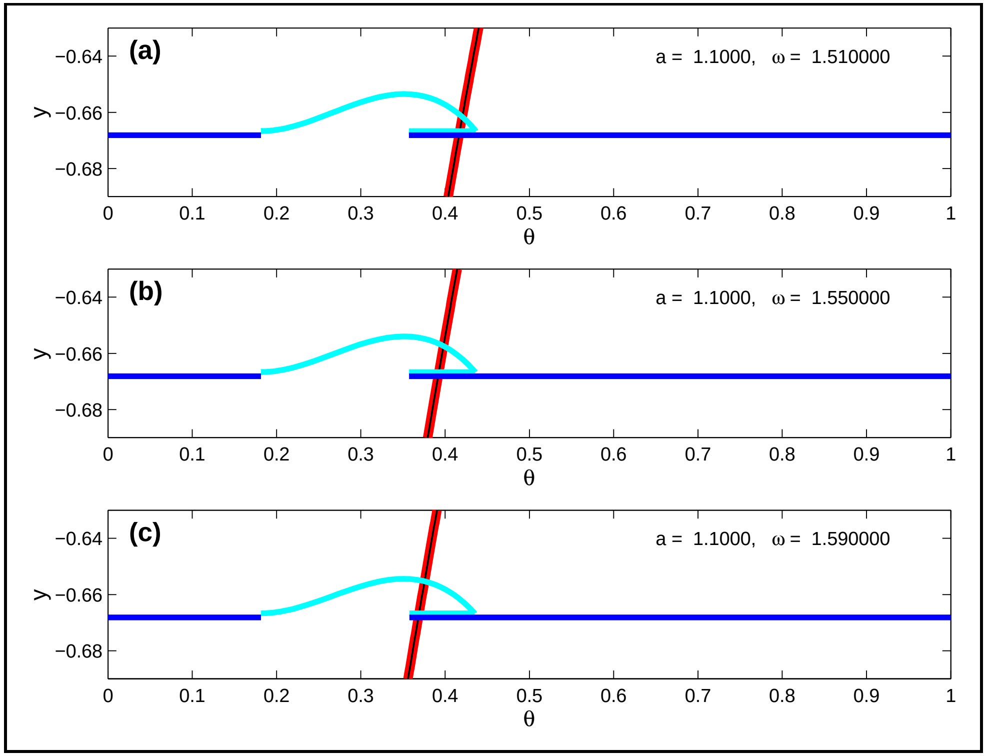

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- Many programs were written... can we automate this?

Reduced System Computing

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- ❑ Canards
- ❑ Fast periodic orbits (and more general ω limit sets of the fast subsystem)

Slow Subsystem - A Few More Details

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Desingularized slow equations:

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Fold points:

$$f(x,y) = 0, \quad \det(D_x f) = 0$$

Fold points are saddle-node equilibria of the fast subsystem.

Example: Two Coupled Neurons

$$\dot{v_1} = -v_1 + \tanh(\sigma_1 v_1) - q_1 - \omega f(v_2)(v_1 + 4)$$

$$\dot{v_2} = -v_2 + \tanh(\sigma_2 v_2) - q_2 - \omega f(v_1)(v_2 + 4)$$

$$\dot{q_1} = \varepsilon(-q_1 + v_1)$$

$$\dot{q_2} = \varepsilon(-q_2 + v_2)$$

$$f(v) = \frac{1}{1 + e^{-40(v - 1/75)}}$$

- Each (v_i, q_i) is a relaxation oscillator.
- When one is firing, the other's v nullcline is depressed ("reciprocal inhibition").
- This system has two fast variables and a two dimensional critical manifold.

Fast/Slow System Definition File: `cpldosc.fs`

```
# Definitions for the coupled oscillator fast/slow system.
cpldosc
# Fast variables:  v1, v2
2
v1
v2
# Slow variables:  q1, q2
2
q1
q2
# Parameters:
3
omega
sigma1
sigma2
# Vector field for the fast variables
-v1+tanh(sigma1*v1) - q1 - omega*(v1+4)/(1+exp(-40*(v2-1/75)))
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Input file: `cpldosc.fs`

Output files:

`fs_cpldosc.c`	General C functions
`fs_cpldosc_cvode.c`	C functions for CVODE
`fs_cpldosc_ida.c`	C functions for IDA
`fs_cpldosc.m`	MATLAB functions

Computer Code Generated

The generated code includes:

Computer Code Generated

The generated code includes:

- *Fast vector field and IVP solver*

Fast vector field, to be used with CVODE

```
int cpldoscF_cv(realtype t, N_Vector Xvec, N_Vector Xdotvec,
                                                      void *params)
```

Fast vector field Jacobian

```
int cpldoscFx_cv(long int N, DenseMat Fx, realtype t,
                        N_Vector Xvec, N_Vector Fvec, void *params,
                       N_Vector tmp1, N_Vector tmp2, N_Vector tmp3)
```

Solve the fast subsystem IVP

```
void cpldosc_fast(FILE *fastfile, double X[], double Y[],
                      double params[], SolverParams *solver_params)
```

Computer Code Generated

The generated code includes:

- *Slow subsystem DAE, configured for IDA*

Compute residuals for the IDA DAE solver

```
int cpldosc_idares(realtype t, N_Vector Zvec, N_Vector Zdotvec,
                                       N_Vector rvec, void *params)
```

Jacobian for IDA DAE solver

```
int cpldosc_idajac(long int Neq, realtype t, N_Vector Zvec,
                                   N_Vector Zdotvec, N_Vector rvec,
                       realtype c_j, void *params, DenseMat jacmat,
                       N_Vector tmp1, N_Vector tmp2, N_Vector tmp3)
```

Solve the slow subsystem DAE IVP

```
void cpldosc_slow_dae(FILE *slowfile, double X[], double Y[],
                      double params[], SolverParams *solver_params)
```

Computer Code Generated

The generated code includes:

- *Desingularized slow subsystem (for CVODE)*

  Desingularized slow subsystem vector field

  ```
  int cpldosc_SlowDE(realtype t, N_Vector Zvec, N_Vector Zdotvec,
                                                        void *params)
  ```

  Desingularized slow subsystem IVP solver

  ```
  void cpldosc_slow_des(FILE *slowfile, double X[], double Y[],
                        double params[], SolverParams *solver_params)
  ```

Computer Code Generated

The generated code includes:

- *Fold function and Jacobian*

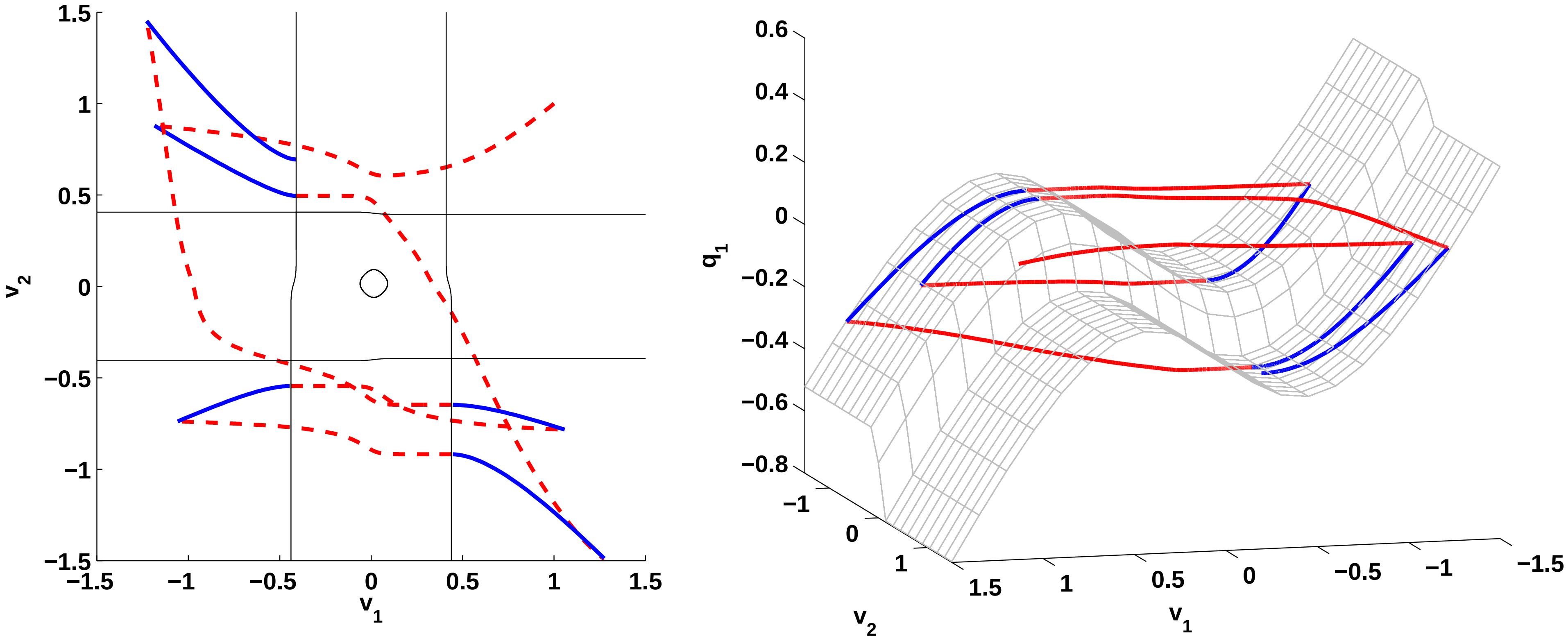
  Fold function

  ```
  realtype cpldosc_foldfunc_nv(N_Vector Zvec, void *params)
  ```

  Fold function gradient (with respect to all variables)

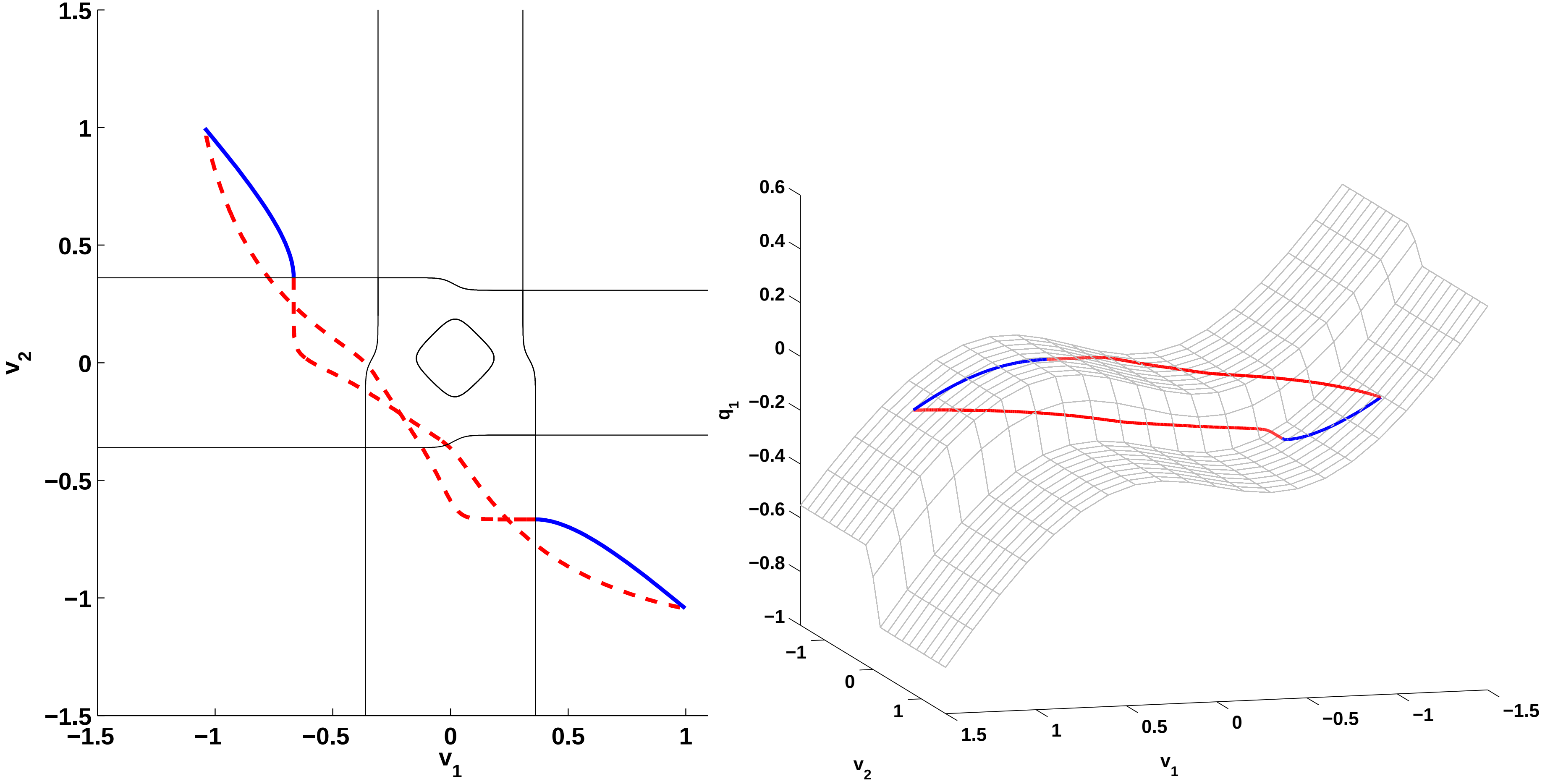
  ```
  void cpldosc_foldfunc_grad_nv(double *grad, N_Vector Zvec,
                                                        void *params)
  ```

Example: Initial Value Problem for the Coupled Oscillator System



$(\omega = 0.05,\ \sigma_1 = 1.5,\ \sigma_2 = 2.6)$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



$\sigma_1 = 1.2$

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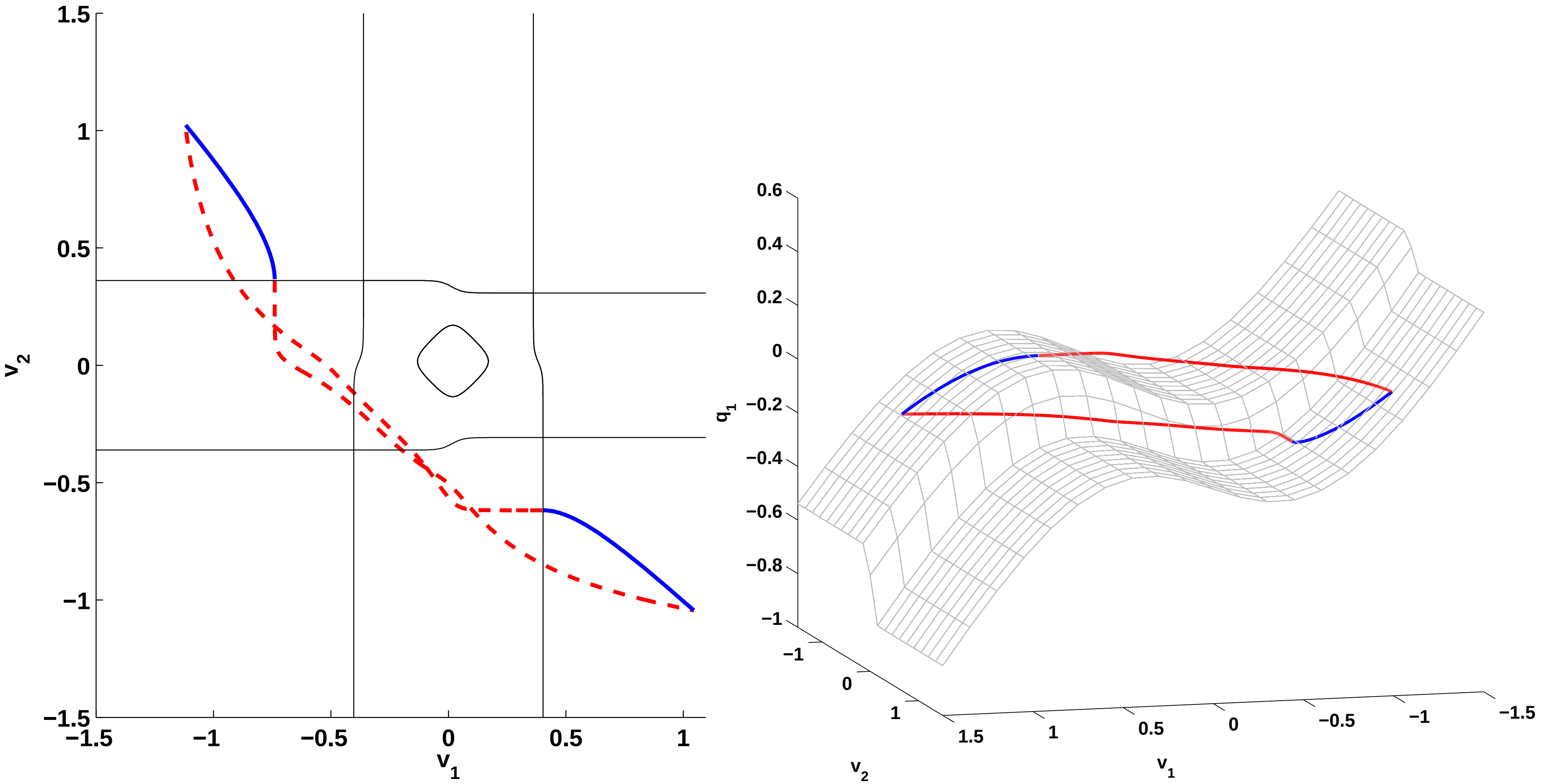

$\sigma_1 = 1.3$

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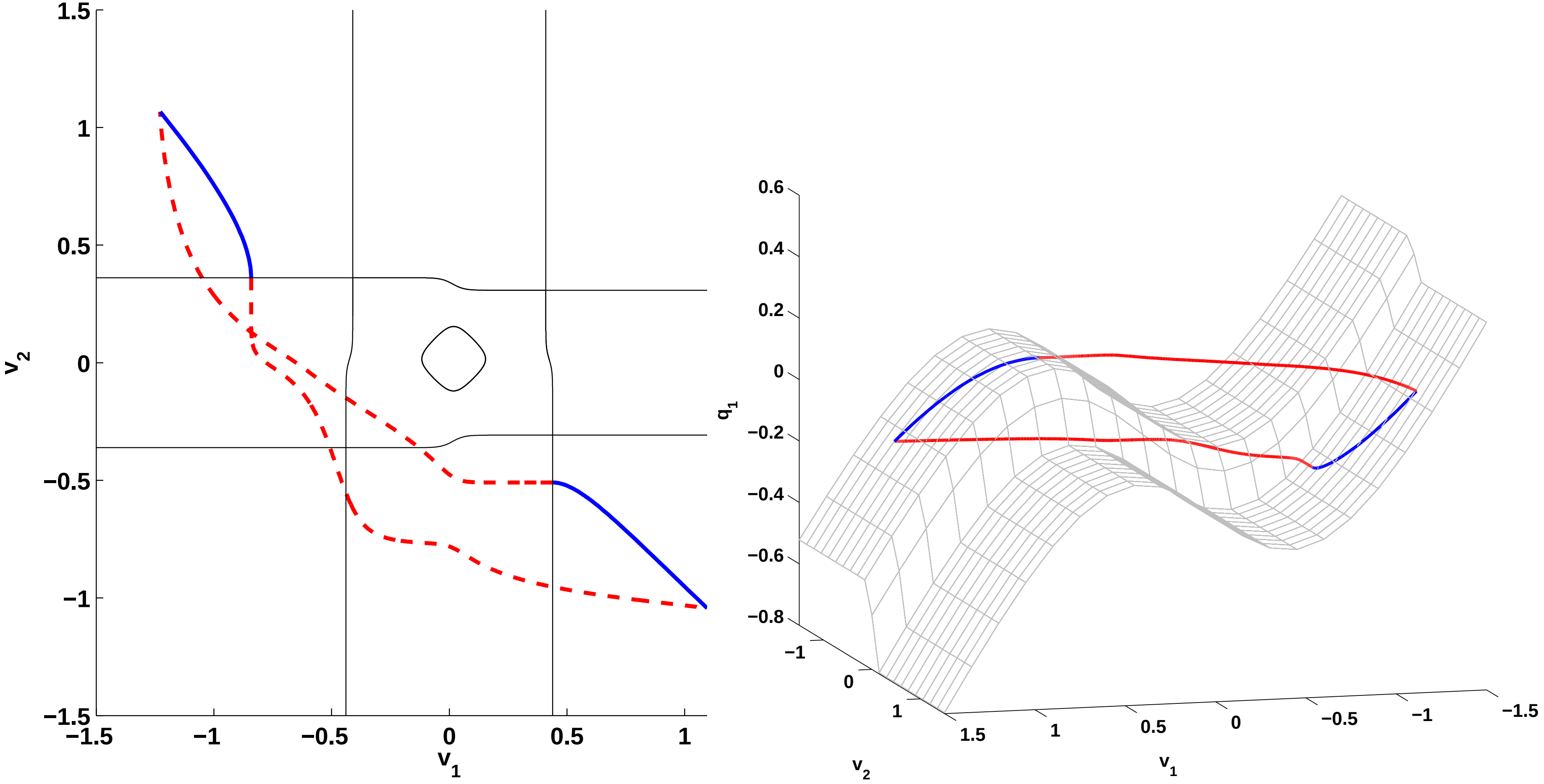
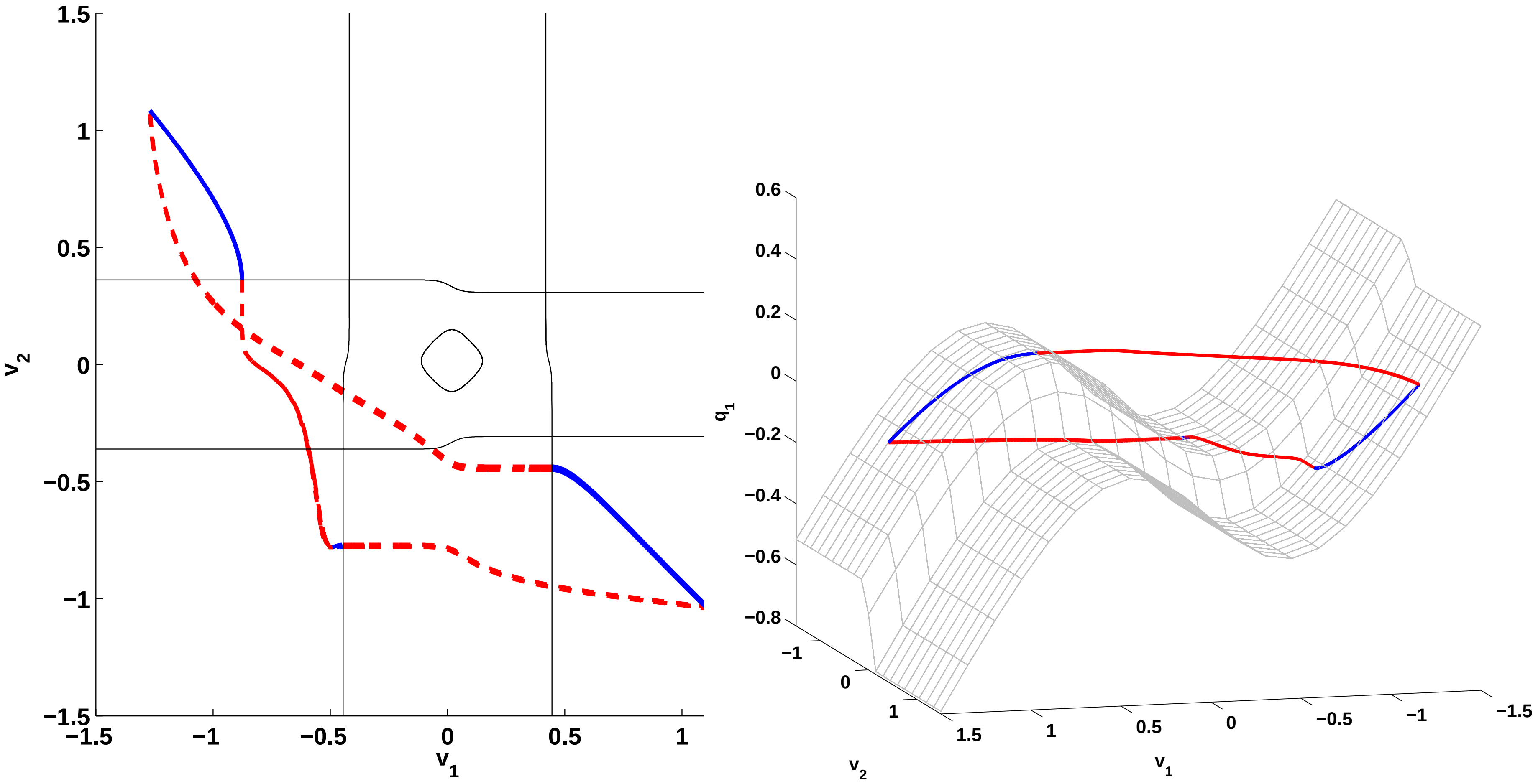


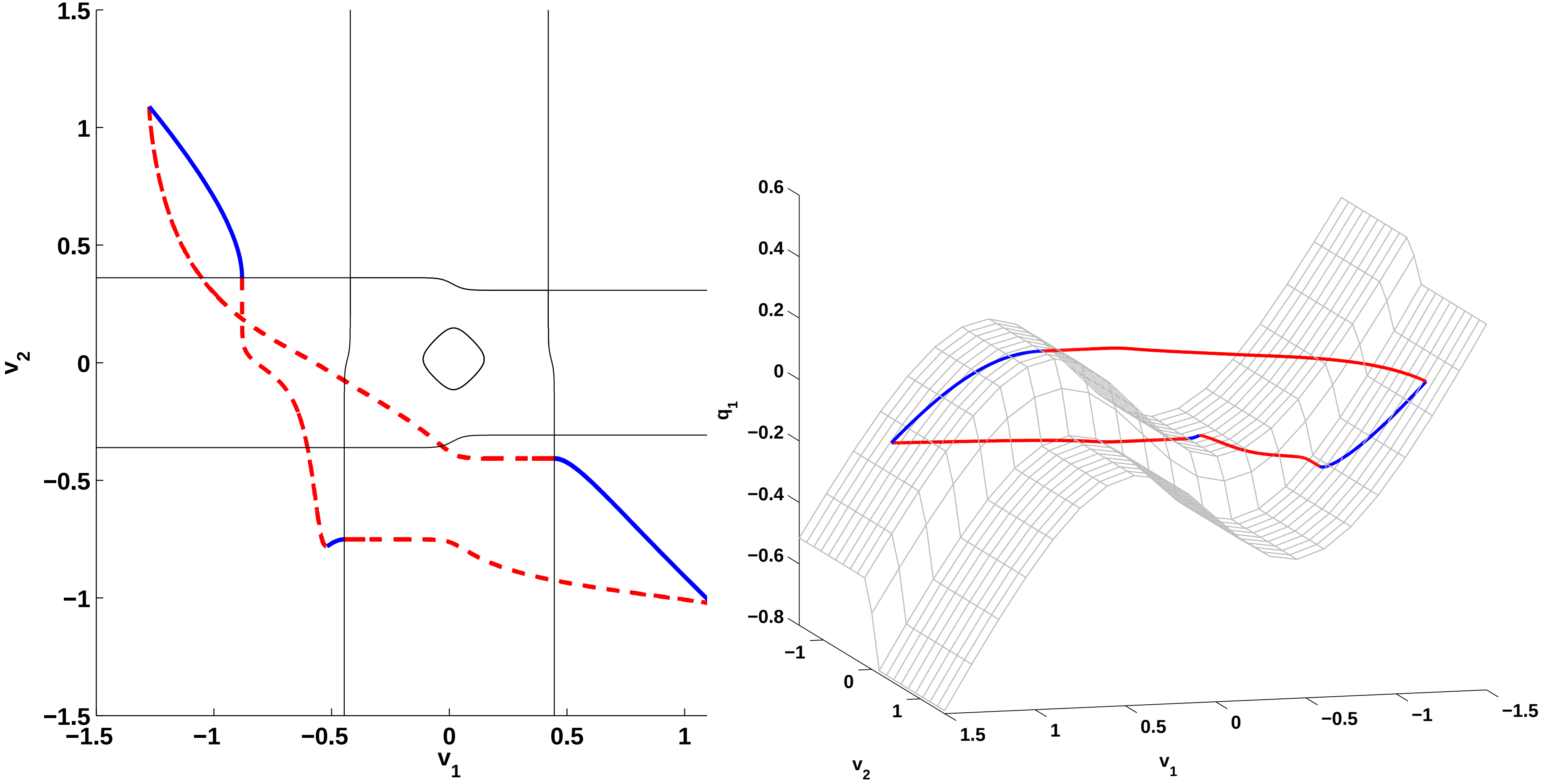
$$\sigma_1 = 1.4$$

Periodic Orbits of the Coupled Oscillator System $(\omega = 0.05,\ \sigma_2 = 1.2)$



$$\sigma_1 = 1.5925$$

Periodic Orbits of the Coupled Oscillator System ($\omega = 0.05$, $\sigma_2 = 1.2$)



$$\sigma_1 = 1.6125$$

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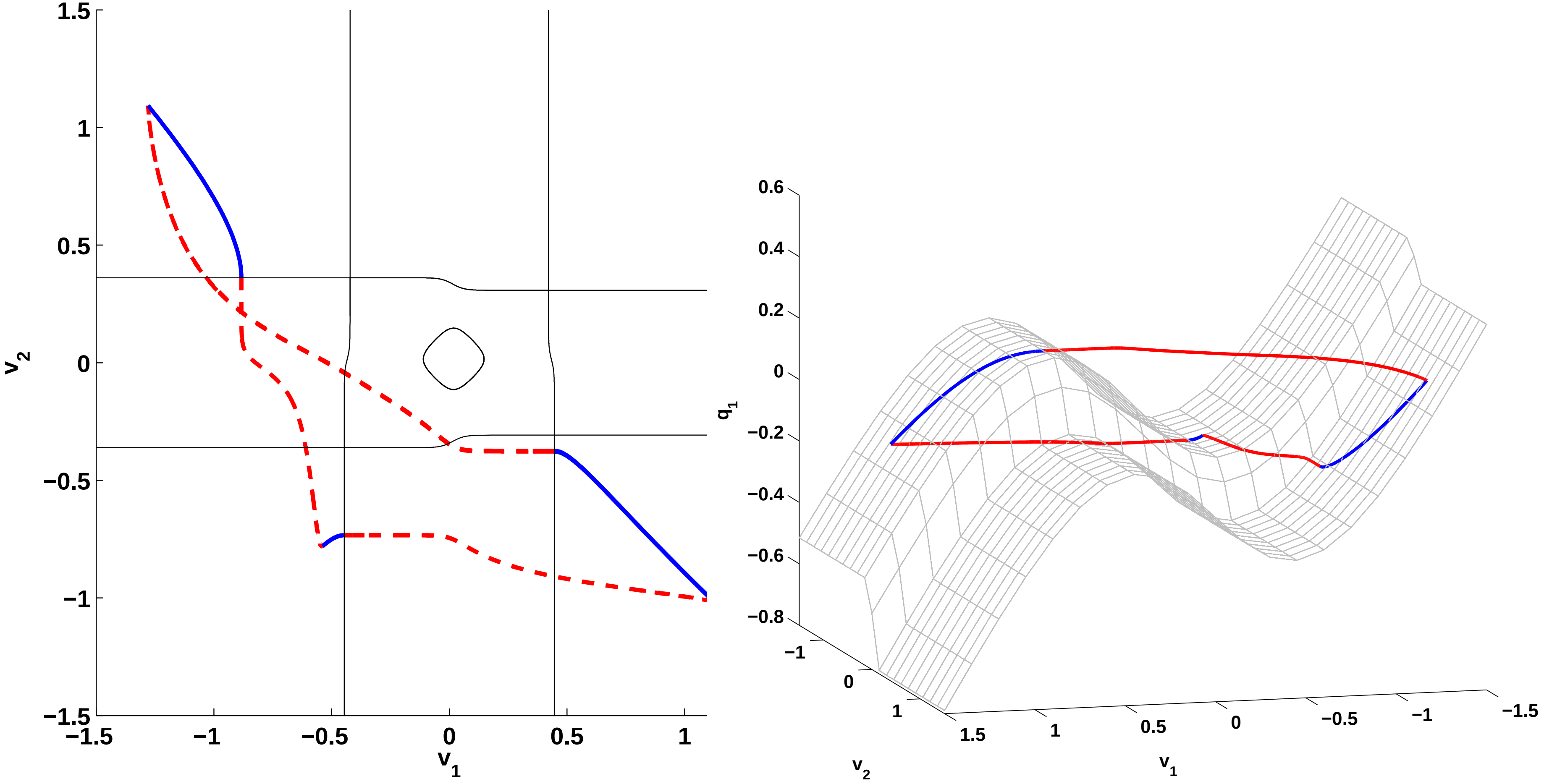

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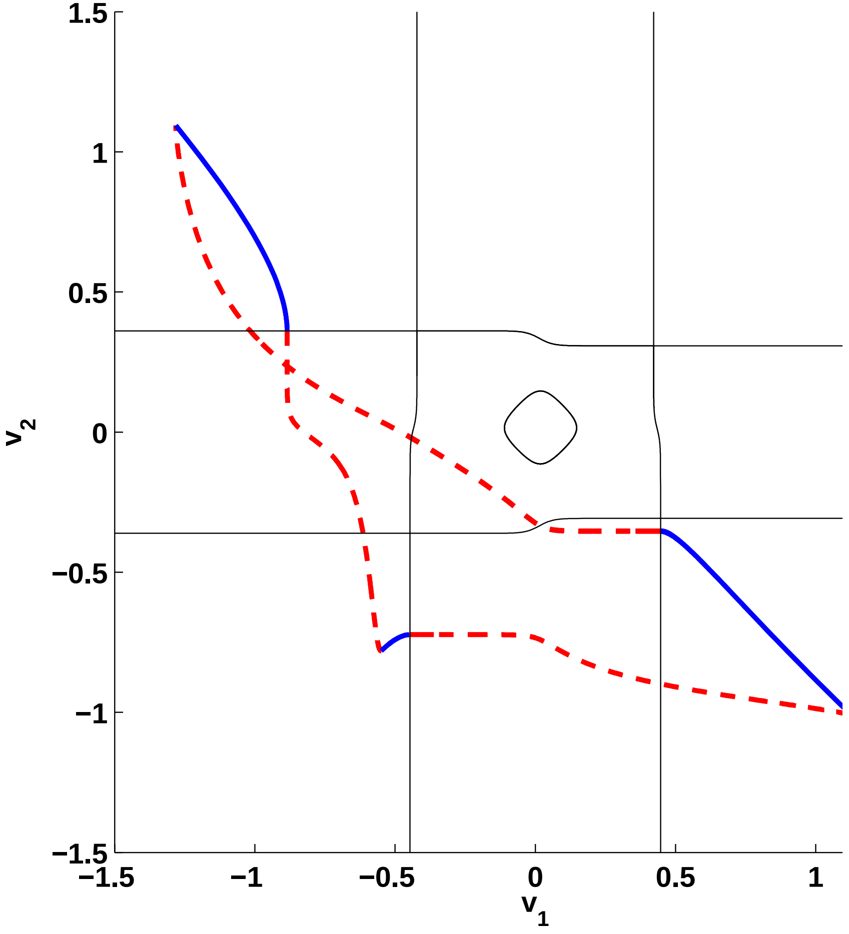

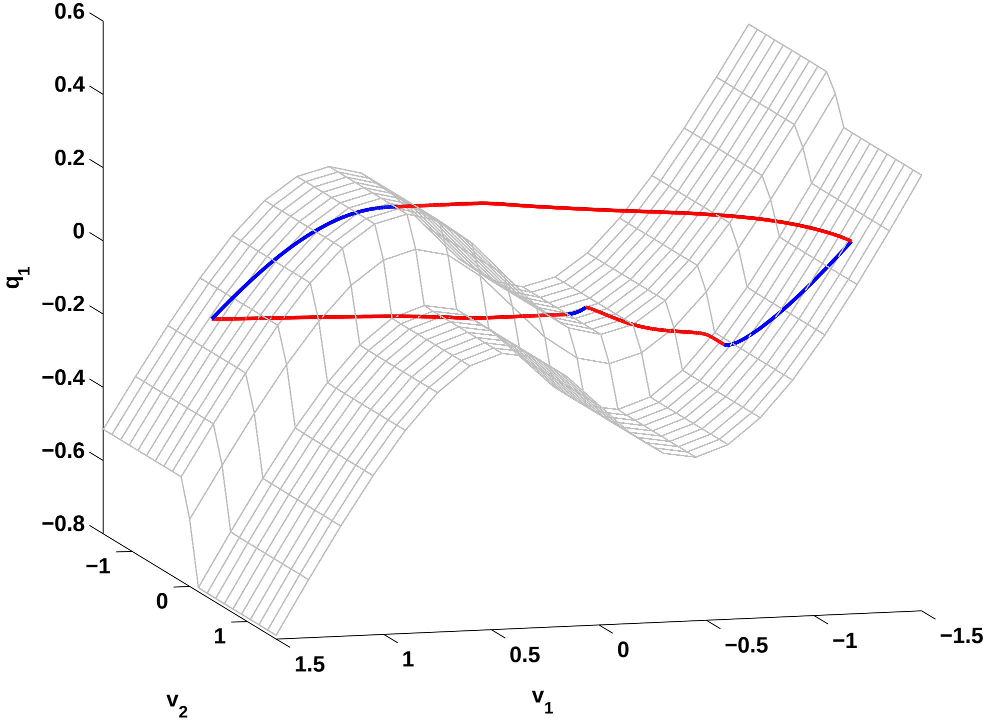

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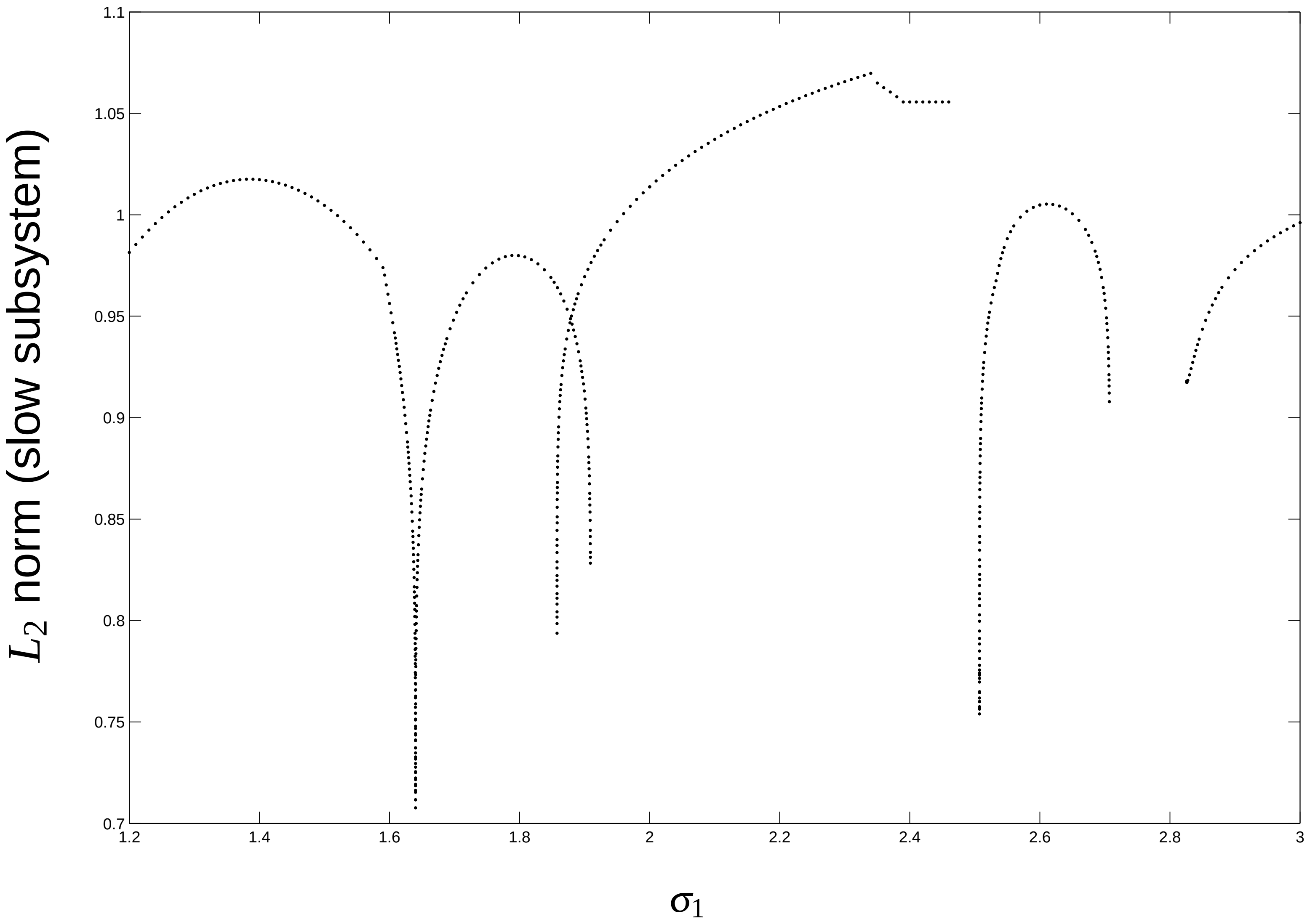
Bifurcation Diagram: Stable Periodic Orbits

(“Poor man’s” continuation; not a complete diagram)



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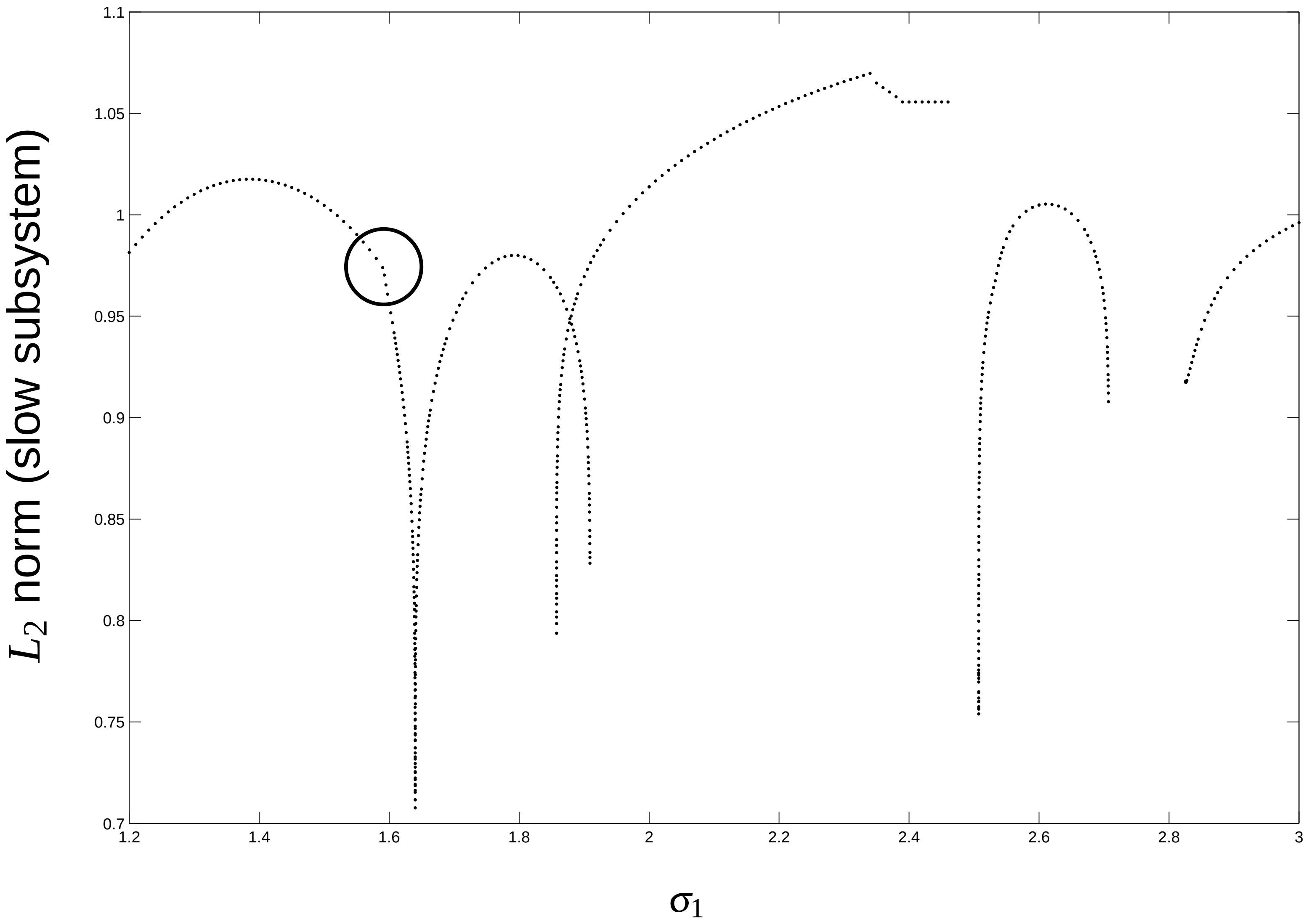

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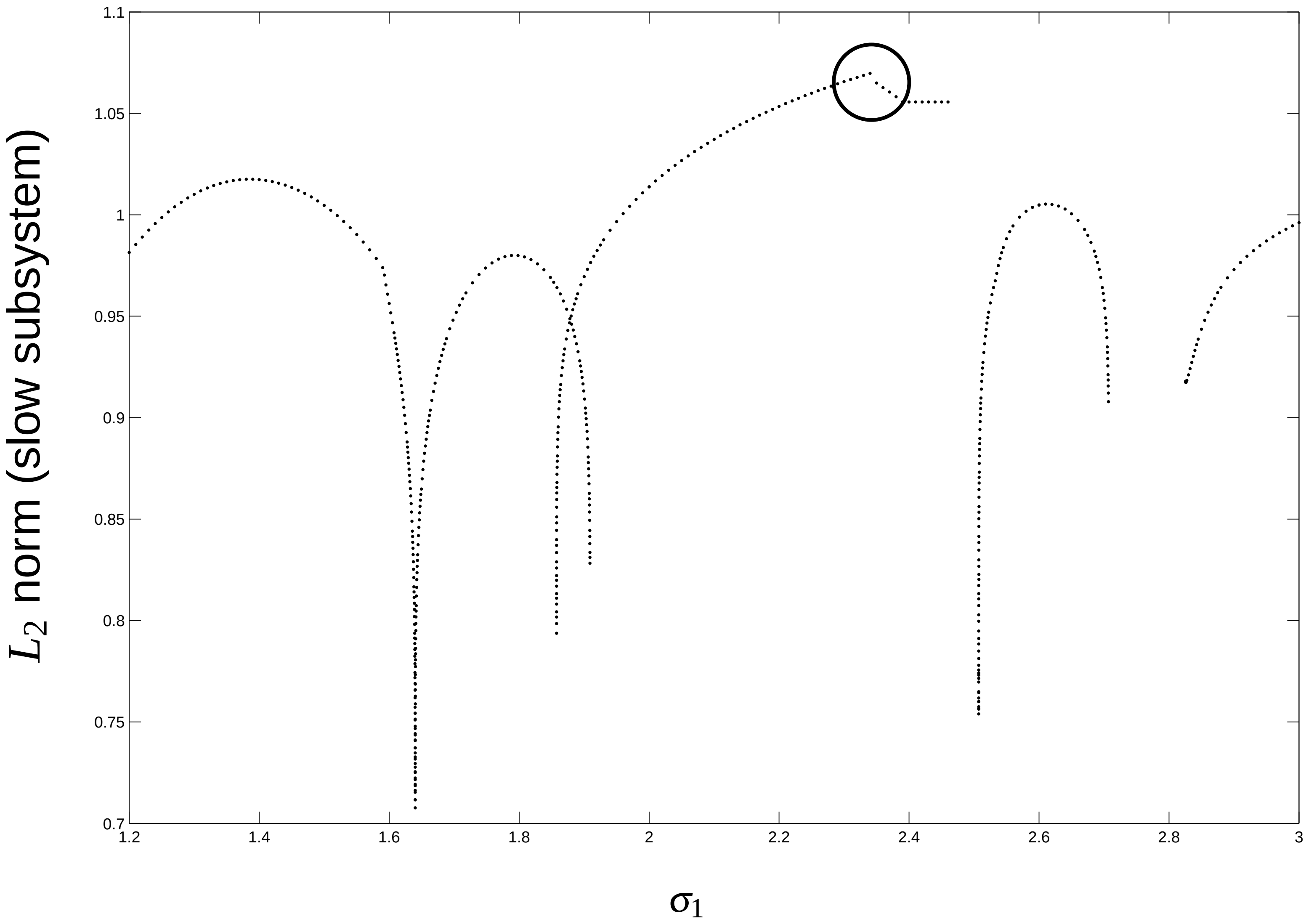

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