Kepler’s First Law of Planetary Motion (see Stewart, pp. 876-7)

(0) The physical assumptions:

\[
\vec{F}(t) = (m)\vec{a}(t) \quad : \text{Newton’s Second Law of Motion}
\]

\[
\vec{F}(t) = \left(\frac{-GmM}{\rho^3}\right)\vec{r}(t) \quad : \text{Law of Gravitation}
\]

where \( m \) and \( M \) are masses, \( G \) is the universal gravitational constant, and \( \rho = \rho(t) = |\vec{r}(t)| \).

In our development, we’ll be considering the motion of a planet around the sun, where we assume the sun determines the origin of our vector representations for position of the planet.

(1) Equating the above, we have

\[
\vec{a}(t) = \left(\frac{-GM}{\rho^3}\right)\vec{r}(t),
\]

and note that the acceleration vector always points in the direction opposite to that of the position vector. Thus \( \vec{r}(t) \times \vec{a}(t) = 0 \).

(2) With this

\[
\frac{d}{dt} (\vec{r}(t) \times \vec{v}(t)) = \vec{v}(t) \times \vec{v}(t) + \vec{r}(t) \times \vec{a}(t) = \vec{v}(t) \times \vec{v}(t) = 0.
\]

So \( \vec{r}(t) \times \vec{v}(t) = \vec{h} \) is a constant vector and \( \vec{h} = 0 \) since the planets aren’t crashing into the sun or moving away.

Now, for any \( t \), \( \vec{r}(t) \) is perpendicular to the constant vector \( \vec{h} \), and we see that all of the motion takes place in a plane.
(3) Let \( \vec{u}(t) = \frac{\vec{r}(t)}{\rho(t)} \) : the unit vector in the direction of the position vector for any \( t \). With this, \( \vec{r}(t) = \rho \vec{u}(t) = \rho(t)\vec{u}(t) \).

(4) Next:

\[
\vec{h} = \vec{r}(t) \times \vec{v}(t) = \vec{r}(t) \times \vec{r}'(t) = (\rho(t)\vec{u}(t)) \times (\rho(t)\vec{u}(t))' \\
= (\rho(t)\vec{u}(t)) \times (\rho'(t)\vec{u}(t) + \rho(t)u'(t)) \\
= \rho^2(t)(\vec{u}(t) \times \vec{u}'(t)) + \rho(t)\rho'(t)(\vec{u}(t) \times \vec{u}(t)) = \rho^2(t)(\vec{u}(t) \times \vec{u}'(t)).
\]

(5) Now,

\[
\vec{a}(t) \times \vec{h} = \left( \frac{-GM}{\rho^2} \right) \vec{u}(t) \times (\rho^2(\vec{u}(t) \times \vec{u}'(t))) = (-GM)\vec{u}(t) \times (\vec{u}(t) \times \vec{u}'(t)) \\
= -GM[(\vec{u}(t) \cdot \vec{u}'(t))\vec{u}(t) - (\vec{u}(t) \cdot \vec{u}(t))\vec{u}'(t)] = (GM)\vec{u}'(t)
\]

using Thm 8.6 (p. 818) and the fact that \( \vec{u}(t) \cdot \vec{u}'(t) = 0 \) since \( |\vec{u}(t)| = 1 \) (see Example 3, class notes from Section 13.2).

(6) So, \( \frac{d}{dt}(\vec{v}(t) \times \vec{h}) = \vec{v}'(t) \times \vec{h} = \vec{a}(t) \times \vec{h} = (GM)\vec{u}'(t) \), and integrating gives

\[\vec{v}(t) \times \vec{h} = (GM)\vec{u}(t) + \vec{c} \text{ for some constant vector, } \vec{c}.\]

(7) Referring to Fig. 8 (p. 877): \( \vec{c} = (\vec{v}(t) \times \vec{h}) - (GM)\vec{u}(t) \) is in the plane perpendicular to \( \vec{h} \) since both vectors making up \( \vec{c} \) have this orientation (see (2)).

Put the standard unit vector \( \vec{k} \) in the direction of \( \vec{h} \) and \( \vec{i} \) in the direction of \( \vec{c} \). Measuring \( \theta \) from \( \vec{i} \), let \( (\rho, \theta) \) denote the polar coordinate of the terminal point of \( \vec{r}(t) \).
(8) Now,
\[
\vec{r}(t) \cdot (\vec{v}(t) \times \vec{h}) = \vec{r}(t) \cdot ((GM)\vec{u}(t) + \vec{c}) = (GM)(\vec{r}(t) \cdot \vec{u}(t)) + (\vec{r}(t) \cdot \vec{c})
\]
\[
= (GM\rho)(\vec{u}(t) \cdot \vec{u}(t)) + (\vec{r}(t) \cdot \vec{c}) = GM\rho + \rho c \cos \theta = \rho(GM + c \cos \theta)
\]
where \( c = |\vec{c}| \). But we also have
\[
\vec{r}(t) \cdot (\vec{v}(t) \times \vec{h}) = (\vec{r}(t) \times \vec{v}(t)) \cdot \vec{h} = \vec{h} \cdot \vec{h} = |\vec{h}|^2 = h.
\]
(Punch line!) So
\[
\rho = \frac{h}{GM + c \cos \theta} = \frac{\left(\frac{h}{GM}\right)}{1 + \left(\frac{c}{GM}\right) \cos \theta}
\]
which we know to describe an \textbf{ellipse} when \( \frac{c}{GM} < 1 \) (and it must be).