An Algorithm for Double Integration via Riemann Sums

Let \( z = z(x, y) \geq 0 \) be defined on the rectangular region \( R = [a, b] \times [c, d] \). The following algorithm defines a process by which we can estimate the volume of the solid bounded below the surface \( z(x, y) \) and above the \( xy \)-plane on \( R \).

1. Subdivide \( R \) into subrectangles of equal area.
   - To do this, subdivide the interval \([a, b]\) up into \( m \) equal subintervals, each having length \( \Delta x = \frac{b-a}{m} \).
   - Now subdivide the interval \([c, d]\) up into \( n \) equal subintervals, each having length \( \Delta y = \frac{b-a}{n} \).
   - This process creates \( mn \) subrectangles each having area \( \Delta A = \Delta x \Delta y \).

2. Choose a point \((x^*, y^*)\) in each of the subrectangles produced by Step 1 (see Figure 2 on handout). In particular,
   - \((x^*_1, y^*_1)\) is a point in the subrectangle determined by the first subinterval in the \( x \)-direction and the first subinterval in the \( y \)-direction - or simply the \((1,1)\)-subrectangle. . .
   - \((x^*_2, y^*_1)\) is a point in the subrectangle determined by the second subinterval in the \( x \)-direction and the first subinterval in the \( y \)-direction - or simply the \((2,1)\)-subrectangle. . .
   - \((x^*_i, y^*_j)\) is a point in the \((i,j)\)-subrectangle. . .
   - \((x^*_m, y^*_n)\) is a point in the \((m,n)\)-subrectangle.

3. Compute the (Riemann) sum
   \[
   z(x^*_1, y^*_1)\Delta x \Delta y + z(x^*_2, y^*_1)\Delta x \Delta y + \ldots + z(x^*_i, y^*_j)\Delta x \Delta y + \ldots + z(x^*_m, y^*_n)\Delta x \Delta y.
   \]
   Note that another, more compact, way to express this sum is
   \[
   \sum_{j=1}^{n} \sum_{i=1}^{m} z(x^*_i, y^*_j) \Delta x \Delta y
   \]
   For convenience, denote this sum by \( V(m,n) \).
4. \( V(m, n) \) is an estimation of the volume bounded below the surface. A better estimate could be obtained by repeating Steps 1 - 3 with larger values of \( m \) and \( n \). This observation leads us to an important definition.

Definition. Let \( z = z(x, y) \) be defined on a rectangular region \( R = [a, b] \times [c, d] \). We say that \( z(x, y) \) is integrable on \( R \) if

\[
\lim_{m, n \to \infty} V(m, n) \text{ exists}
\]

where

\[
V(m, n) = \sum_{j=1}^{n} \sum_{i=1}^{m} z(x_i^*, y_j^*) \Delta x \Delta y
\]

If the limit exists (as a finite number) we write

\[
\int \int_R z(x, y) \, dA
\]

for this number and call it the value of the (double) integral of \( z(x, y) \) on \( R \).

Notes on the process of double integration.

- This algorithm produces a double integral value by way of Riemann sums.
- If ‘partial antiderivatives’ for \( z(x, y) \) can be found, then the double integral value can be computed using the method of iterated integration (i.e. Approach I). But the algorithm described above may work even if an iterative approach fails. (See example where \( z = e^{-x^2-y^2} \).)
- If \( x(x, y) \) is continuous on \( R \), then the process described above is guaranteed to produce a finite value. That is, any continuous function of two variables is (doubly) integrable. There are even some discontinuous functions for which the double integral value can be computed!
- It is not necessary that \( z(x, y) \geq 0 \) on \( R \) in order to compute the value of the double integral. On the other hand, if this is true, then the value of the double integral will measure the volume of the solid bounded below the surface \( z(x, y) \) and above the \( xy \)-plane on \( R \).