Triple Integration over Solids

We are now interested in computing triple integrals over solids other than (cartesian) boxes. Let $E$ denote the solid over which our function $f(x, y, z)$ is defined.

**Solid Type I.** This type of solid is characterized by the fact that $z$ coordinates are bounded between two functions of $x$ and $y$. That is,

$$z_1(x, y) \leq z \leq z_2(x, y)$$

for all points within $E$. (See the figures for this section.)

For such a solid

$$\int \int \int_E f(x, y, z) dV = \int \int_D \left[ \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz \right] dA$$

where $D$ is the region obtained by projecting $E$ onto the $xy$-plane. Note after the first integration (in the $z$-direction) the problem is essentially that of double integration over $D$. In particular . . .

- $D$ is a Type 1 region in the $xy$-plane. In this case

$$\int \int \int_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz dy dx.$$  

- $D$ is a Type 2 region in the $xy$-plane. In this case

$$\int \int \int_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{z_1(x, y)}^{z_2(x, y)} f(x, y, z) dz dx dy.$$  

- $D$ is a Type 3 region in the $xy$-plane. In this case

$$\int \int \int_E f(x, y, z) dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{z_1(r\theta)}^{z_2(r\theta)} f(r, \theta, z) r dz dr d\theta.$$  

**Note** We are actually integrating by characterizing $E$ in the cylindrical coordinate system in this case.
• **D** is a Type 4 region in the $xy$-plane. In this case

$$
\int \int \int_E f(x, y, z)dV = \int_a^b \int_{h_1(r)}^{h_2(r)} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r, \theta, z) r \, dz \, d\theta \, dr.
$$

Again we are integrating by characterizing $E$ in the cylindrical coordinate system.

**Solid Type II.** This type of solid is characterized by the fact that $x$ coordinates are bounded between two functions of $y$ and $z$. That is,

$$
x_1(y, z) \leq x \leq x_2(y, z)
$$

for all points within the solid, $E$. (See the section figures).

For such a solid

$$
\int \int \int_E f(x, y, z)dV = \int \int_D \left[ \int_{x_1(y, z)}^{x_2(y, z)} f(x, y, z) \, dx \right] dA
$$

where $D$ is the region obtained by projecting $E$ onto the $yz$-plane.

Now $D$ can be one of the four possible types of regions that we have discussed making the appropriate change in variables to account for the fact that this 'underlying' domain lies in the $yz$-plane.

**Solid Type III.** This type of solid is characterized by the fact that $y$ coordinates are bounded between two functions of $x$ and $z$. That is

$$
y_1(x, z) \leq y \leq y_2(x, z)
$$

for all points within the solid, $E$. See the section figures.

For such a solid

$$
\int \int \int_E f(x, y, z)dV = \int \int_D \left[ \int_{y_1(x, z)}^{y_2(x, z)} f(x, y, z) \, dy \right] dA
$$

where $D$ is the region obtained by projecting $E$ onto the $xz$-plane.

Now $D$ can be one of the four possible types of regions that we have discussed making the appropriate change in variables to account for the fact that this 'underlying' domain lies in the $xz$-plane.