2.8) Let $G$ be the set of all real-valued functions $f$ on the real line which have the property that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Define the product $f \times g$ of two functions $f, g$ in $G$ by $
abla f \times g \times (x) = f(x) g(x)$ for all $x \in \mathbb{R}$.

With this operation, does $G$ form a group? Prove or disprove.

Yes, we will show that $(G, \times)$ satisfies all four conditions of being a group.

i) $G$ is a set and $\times$ is a binary operation on $G$

$G$ is defined as a set.

$(f \times g) (x) = f(x) g(x)$ is a well-defined function on $\mathbb{R}$.

Thus, to show $f \times g \in G$ we must verify that $(f \times g) (x) \neq 0$ for all $x \in \mathbb{R}$.

Assume otherwise, that is, there is an $x_0 \in \mathbb{R}$ such that $(f \times g) (x_0) = f(x_0) g(x_0) = 0$

Thus, $f(x_0) = 0$ or $g(x_0) = 0$, contradicting $f, g \in G$.

ii) $\times$ is associative

$((f \times g) \times h) (x) = (f(x) g(x)) h(x) = f(x) (g(x) h(x)) = (f \times (g \times h)) (x)$

This fact follows from the associativity of real numbers under multiplication.

iii) There is an identity element $e$ where $e \times f = f \times e = f$ for any $f \in G$

Define $e(x) = 1$; for any $g \in G$

$(g \times e) (x) = g(x) \cdot 1 = g(x) = 1 \cdot g(x) = (e \times g) (x)$

iv) For each $f \in G$ there is an $f^{-1}$ such that $f \times f^{-1} = f^{-1} \times f = e$

Define $f^{-1}(x) = \frac{1}{f(x)}$ for all $x \in \mathbb{R}$. $f^{-1}$ is well defined on $\mathbb{R}$ because $0 \notin f(\mathbb{R})$

$(f^{-1} \times f) (x) = f(x) \frac{1}{f(x)} = 1 = e = \frac{1}{f(x)} f(x) = (f \times f^{-1})(x)$