Math 111 E and H — Exam I

Show all work clearly for partial credit. Do not use the graphing capabilities of your calculator.

1. (8 points) Find exactly; i.e., do not use a calculator:
   
   \( \text{a) } \) \( \log_3 45 - \log_3 5 \) \( \text{b) } \) \( \frac{\log_2 \sqrt{27}}{\log_2 3} \)

2. (20 points) Find the limits, if they exist:

   \( \text{a) } \) \( \lim_{x \to 3} \frac{x^2 - 3x + 2}{x^2 - 4} \) \( \text{b) } \) \( \lim_{x \to 2} \frac{x^2 + 3x + 2}{x^2 - 4} \)

   \( \text{c) } \) \( \lim_{x \to -\infty} (2e^{-x} - 1) \) \( \text{d) } \) \( \lim_{x \to 0^-} \left( \frac{1}{x} - \frac{1}{|x|} \right) \)

3. (14 points) For the function \( f(x) \) with the graph below, find or approximate (if they exist):

   \( \text{a) } \) \( \lim_{x \to -2^-} f(x) \),
   
   \( \text{b) } \) the equation(s) of the vertical asymptote(s),
   
   \( \text{c) } \) the equation(s) of the horizontal asymptote(s),
   
   \( \text{d) } \) \( \lim_{x \to 0} f(x) \),
   
   \( \text{e) } \) the \( x \)-value(s) at which \( f \) has a removable discontinuity,
   
   \( \text{f) } \) \( f'(-1) \) and
   
   \( \text{g) } \) \( f'(1) \)

4. (18 points) Find the equations of the horizontal and vertical asymptotes:

   \( \text{a) } \) \( f(x) = \frac{x^2 - 1}{x^2 - 3x + 2} \) \( \text{b) } \) \( g(x) = \frac{x}{\sqrt{x^2 + 1}} \)
5. (15 points) (a) Use the limit definition of the derivative (in either form) to find \( f'(11) \), the derivative of \( f(x) = \sqrt{2x + 3} \) at \( a = 11 \). Use of a derivative formula or simply the answer will receive no credit.

(b) Write the equation of the tangent line to \( y = \sqrt{2x + 3} \) at \( x = 11 \).

6. (15 points) Let \( f(x) = x^2 \), \( a = 2 \) and \( \varepsilon = 0.5 \). What is the largest value of \( \delta \) for which, if \( 0 < |x - 2| < \delta \), then \( |x^2 - 4| < 0.5 \)?

(An answer of the form “the smaller of the two numbers _______ and _______” is preferred but not required.)

6. (18 points) Which of the lettered graphs A, B or C is that of the derivative of \( f(x) \)? Explain your answer.

Some possibly useful equations:

\[
\begin{align*}
y - y_0 &= m(x - x_0) \\
a^3 - b^3 &= (a - b)(a^2 + ab + b^2) \\
a^r &= b \iff \log_a b = r \\
\log_a c &= \frac{\log_b c}{\log_b a}
\end{align*}
\]
Solutions to Math 111EH Exam I

1. (a) \(\log_3\left(\frac{45}{5}\right) = \log_3 9 = 2\).
   (b) \(\log_3 \sqrt[3]{27} = \log_3 (3^{3/2}) = \frac{3}{2}\).

2. (a) \(\lim_{x \to 2} \frac{(x-2)(x-1)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x-1}{x+2} = \frac{2-1}{2+2} = \frac{1}{4}\).
   (b) \(\lim_{x \to 2} \frac{(x+2)(x+1)}{(x-2)(x+2)} = \lim_{x \to 2} \frac{x+1}{x-2}\), which does not exist: as \(x\) approaches 2 from the left, the quotient approaches \(-\infty\), while as \(x\) approaches 2 from the right, the quotient approaches \(\infty\).
   (c) As \(x\) grows without bound, \(-x\) approaches \(-\infty\), so \(e^x\) approaches 0; and hence the required limit is 2(0) = 0.
   (d) For \(x\)-values less than 0, we have \(|x| = -x\), so \((1/x) - (1/|x|) = 2/x\), which approaches \(-\infty\) as \(x\) approaches 0 from the left.

3. (a) 1 (as nearly as we can tell).
   (b) \(x = -2\) and \(x = 0\) (the y-axis).
   (c) \(y = 1\) to the left and \(y = 0\) (the x-axis) to the right.
   (d) The limit does not exist. The two one-sided limits are \(\infty\) and \(-\infty\), so they do not agree.
   (e) \(x = 3\). The rest are jump discontinuities or worse.
   (f) 0 and (g) about 1.

4. (a) \(\lim_{x \to \infty} \frac{1 - (1/x^2)}{(1 - (3/x) + (2/x^2))} = (1 - 0)/(1 - 0 + 0) = 1\), and this is also the limit as \(x \to -\infty\), so \(y = 1\) is a horizontal asymptote to both the right and left. Because \((x^2 - l)/(x^2 - 3x + 2) = (x + l)/(x - 2)\) except at \(x = 1\), there is a removable discontinuity at \(x = 1\); the only vertical asymptote is \(x = 2\).
   (b) Because the denominator is never 0, there are no vertical asymptotes. Now:
   \[
   \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to \infty} \frac{1}{\sqrt{1 + (1/x^2)}} = \frac{1}{\sqrt{1 + 0}} = 1
   \]
   \[
   \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x \to -\infty} \frac{1}{-\sqrt{1 + (1/x^2)}} = -\frac{1}{\sqrt{1 + 0}} = -1
   \]
   So \(y = 1\) is a horizontal asymptote to the right and \(y = -1\) is one to the left.

5. (a) \(f'(11) = \lim_{h \to 0} \frac{\sqrt{2(11+h) + 3} - \sqrt{2(11) + 3}}{h} = \lim_{h \to 0} \frac{2(2(11 + h) + 3) - (2(11) + 3)}{h(\sqrt{2(11+h) + 3} + \sqrt{2(11) + 3})}
   = \lim_{h \to 0} \frac{2}{\sqrt{2(11+h) + 3} + \sqrt{2(11) + 3}} = \frac{2}{\sqrt{2(11) + 3}} = \frac{1}{5}\)
   (b) Because \(f(11) = 5\), the desired tangent line is \(y - 5 = \frac{1}{5}(x - 11)\).

6. Because the inverse of the function \(y = x^2\) is \(x = \sqrt{y}\), at least on the part of the curve in which we are interested, the answer is: the largest \(\delta\) that works is the smaller of the two numbers \(2 - \sqrt{3.5}\) and \(\sqrt{3.5} - 2\).

7. C: The tangents to \(f(x)\) have small negative slopes both to the left and to the right, and in between the slopes become positive, with the greatest slope at \(x = 0\).