Math 112 — Practice Exam I

Show all work clearly; an answer with no justifying computations may not receive credit (except in the “set up but do not evaluate” problems).

1. (15 points) (a) Find the equation of the two tangent lines to the graph of \( x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \) at the points \((0, \pm \frac{1}{2})\).

   (b) Find \( \frac{d}{dx} (\arcsin(2x)) \).

2. (21 points) Find the following limits:

   \[
   \begin{align*}
   \text{(a) } & \lim_{x \to \infty} (x - \sqrt{x^2 - 3x}) \\
   \text{(b) } & \lim_{x \to 0} x \csc x \\
   \text{(c) } & \lim_{x \to \infty} (1 - \frac{1}{x})^x
   \end{align*}
   \]

3. (18 points) Set up but do not evaluate a definite integral that represents the volume of each of the following solids:

   (a) The base is the region bounded by the \( x \)-axis, the line \( x = 1 \) and the parabola \( y = x^2 \); the cross sections perpendicular to the \( x \)-axis are equilateral triangles.

   (b) The solid of rotation generated by rotating about the \( x \)-axis the region above the \( x \)-axis and under \( y = 3(2 - x) \ln x \).

   (c) The solid of rotation generated by rotating the same region as in (c) about the line \( x = 3 \).

   \[
   \begin{align*}
   \text{(a)} \quad \text{Region to be rotated:} \\
   \text{(b) and (c) Region to be rotated:}
   \end{align*}
   \]
4. *(10 points)* A liquid weighing 9800 N/m\(^3\) fills a tank in the shape below. Set up but *do not evaluate* a definite integral that gives the work done in emptying the tank. Note: The width of the tank is proportional to the distance from the bottom of the tank.

![Diagram of a tank](image)

5. *(15 points)* Find the antiderivative \(\int (x^2 + 9) \ln x \, dx\) by integration by parts.

6. *(21 points)* Evaluate the following integrals:

   (a) \(\int \sec^3 x \tan^4 x \, dx\)  
   (b) \(\int \cos 2x \sin^4 x \, dx\)  
   (c) \(\int \frac{\cos x}{1 + \sin^2 x} \, dx\)

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Some possibly useful formulas:

\[
\begin{align*}
\sin A \cos B &= \frac{1}{2}(\sin(A + B) + \sin(A - B)) \\
\sin A \sin B &= \frac{1}{2}(\cos(A - B) - \cos(A + B)) \\
\cos A \cos B &= \frac{1}{2}(\cos(A - B) + \cos(A + B)) \\
\sin 2A &= 2 \sin A \cos A \\
\cos 2A &= \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \\
\int \csc x \, dx &= \ln |\csc x - \cot x| + C \\
\int \cot x \, dx &= \ln |\sin x| + C \\
y - y_0 &= m(x - x_0)
\end{align*}
\]
Solutions to Exam I

1. Implicit differentiation of the given equation gives \(2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)\), so dividing by 2 and moving all the terms with a \(y'\) to the left side of the equation and all others to the right gives \(y'(y - 4y(2x^2 + 2y^2 - x)) = (4x - 1)(2x^2 + 2y^2 - x) - x\), or

\[
y' = \frac{(4x - 1)(2x^2 + 2y^2 - x) - x}{y - 4y(2x^2 + 2y^2 - x)}.
\]

At the point \((0, \frac{1}{2})\), we get \(y' = [(0 - 1)(0 + \frac{1}{2} - 0)]/[\frac{1}{2} - 2(0 + \frac{1}{2} - 0)] = 1\), so the desired tangent line is \(y - \frac{1}{2} = 1(x - 0)\) (or \(y = \frac{1}{2} + x\)). At the point \((0, -\frac{1}{2})\), we get \(y' = [(0 - 1)(0 + \frac{1}{2} - 0)]/[-\frac{1}{2} + 2(0 + \frac{1}{2} - 0)] = -1\), so the desired tangent line is \(y + \frac{1}{2} = -1(x - 0)\) (or \(y = -\frac{1}{2} - x\)).

(b) \(1/\sqrt{1 - (2x^2)} = 2/\sqrt{1 - 4x^2}\).

2. (a) We multiply numerator and denominator \((1)\) by \(x + \sqrt{x^2 - 3x}\) and divide numerator and denominator by \(x\); then the result is clear:

\[
\lim_{x \to \infty} (x - \sqrt{x^2 - 3x}) = \lim_{x \to \infty} \frac{x^2 - (x^2 - 3x)}{x + \sqrt{x^2 - 3x}} = \lim_{x \to \infty} \frac{3x}{x + \sqrt{x^2 - 3x}} = \lim_{x \to \infty} \frac{3}{1 + \sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1 + 0}} = \frac{3}{2}.
\]

Notice that we didn’t use l’Hospital’s rule; it would be valid (after the first step), but it would not make the limit easier to find.

(b) As \(x \to 0\), \(\csc x\) approaches \(\pm \infty\) (one from the left, the other from the right), so the limit is indeterminate. So we rewrite as a quotient and use l’Hospital’s rule (or just recite the limit that was used in finding the derivative of \(\sin x\):

\[
\lim_{x \to 0} x \csc x = \lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{1} = 1.
\]

(c) Because the form \(1^\infty\) is an exponential indeterminate form, we find the limit of the log of the function:

\[
\lim_{x \to \infty} \ln \left(\left(1 - \frac{1}{x}\right)^x\right) = \lim_{x \to \infty} x \ln \left(1 - \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\ln \left(1 - \frac{1}{x}\right)}{x^{-1}}
\]

\[
= \lim_{x \to \infty} \frac{1 - \frac{1}{x}}{-x^{-2}} = \lim_{x \to \infty} \frac{-1}{1 - \frac{1}{x}} = -\frac{1}{1 - 0} = -1,
\]

so the desired limit is \(e^{-1} = 1/e\).

3. For (a), because the height of an equilateral triangle of side \(s\) is \((\sqrt{3}/2)s\), its area is \((\sqrt{3}/4)s^2\); and because the slice of the solid at a given \(x\)-value is a triangle of side \(x^2\), its area is \((\sqrt{3}/4)x^4\); so

(a) \(\int_0^1 \frac{\sqrt{3}}{4} (x^4)dx\)
(b) \(\int_1^2 \pi(3(2 - x)\ln x)^2dx\)
(c) \(\int_1^2 2\pi(3 - x)(3(2 - x)\ln x)dx\)
4. A slice of liquid that is \( x \) m above the bottom of the tank, of thickness \( dx \), has length 8 m and width \( w \), where \( w/x = 3/5 \), or \( w = (3/5)x \), so its weight in newtons is 9800(8)(3/5)x \( dx \). That slice has to be moved vertically 7 – \( x \) m. So the total amount of work done is

\[
W = \int_{0}^{5} (7-x)9800(8)(3/5)x \, dx .
\]

5. Because the antiderivative of \( \ln x \) is not well-known, let us begin by setting \( u = \ln x \) and \( dv = (x^2 + 9)dx \), so that \( du = (1/x)dx \) and \( v = \frac{1}{3}x^3 + 9x \). Then we get

\[
\int (x^2 + 9)\ln x \, dx = \left( \frac{1}{3}x^3 + 9x \right)\ln x - \int \left( \frac{1}{3}x^3 + 9x \right) \frac{1}{x} \, dx - \frac{1}{9}x^3 - 9x + C .
\]

6. (a) Neither of the obvious substitutions, \( u = \tan x \) or \( u = \sec x \), seems to work, because the coefficients are inconvenient: using the Pythagorean relation between secant and tangent, we can’t conveniently leave two factors of secant or one of tangent. One way to go would be to create a reduction formula for \( \int \tan^m x \sec^n x \, dx \), using integration by parts, with \( u = \tan^{m-1} x \) and \( dv = \sec^n x \tan x \, dx \), so that \( du = (m - 1)\tan^{m-2} x \sec^2 x \, dx \) and \( v = \frac{1}{n} \sec^n x \):

\[
\int \tan^m x \sec^n x \, dx = \frac{1}{n} \tan^{m-1} x \sec^n x - \frac{m-1}{n} \int \tan^{m-2} x \sec^{n+2} x \, dx
= \frac{1}{n} \tan^{m-1} x \sec^n x - \frac{m-1}{n} \int \tan^{m-2} x (\tan^2 x + 1) \sec^n x \, dx
= \frac{1}{n} \tan^{m-1} x \sec^n x - \frac{m-1}{n} \int \tan^m x \sec^n x \, dx
- \frac{m-1}{n} \int \tan^{m-2} x \sec^n x \, dx.
\]

Adding the middle term of the last expression to both ends and dividing both ends by \( (n + m - 1)/n \), we get

\[
\int \tan^m x \sec^n x \, dx = \frac{1}{n+m-1} \tan^{m-1} x \sec^n x - \frac{m-1}{n + m - 1} \int \tan^{m-2} x \sec^n x \, dx.
\]

Using this and the reduction formula on page 481, exercise 48, we get

\[
\int \tan^4 x \sec^3 x \, dx = \frac{1}{6} \tan^3 x \sec^3 x - \frac{1}{2} \int \tan^2 x \sec^3 x \, dx
= \frac{1}{6} \tan^3 x \sec^3 x - \frac{1}{2} \left( \frac{1}{4} \tan x \sec^3 x - \frac{1}{4} \int \sec^3 x \, dx \right)
= \frac{1}{6} \tan^3 x \sec^3 x - \frac{1}{8} \tan x \sec^3 x
+ \frac{1}{8} \left( \frac{1}{2} \tan x \sec x + \frac{1}{2} \int \sec x \, dx \right)
= \frac{1}{6} \tan^3 x \sec^3 x - \frac{1}{8} \tan x \sec^3 x
+ \frac{1}{16} \tan x \sec x + \frac{1}{16} \ln |\sec x + \tan x| + C .
\]

This doesn’t use anything we don’t already know, but it’s complicated. The method suggested in class was to change the integrals into sines and cosines:

\[
\int \frac{1}{\cos^3 x} \cdot \sin^4 x \, dx = \int \frac{\sin^4 x}{\cos^8 x} \cos x \, dx.
\]
Now we can replace the \( \cos^8 x \) in the denominator with \((1 - \sin^2 x)^4\) and make the substitution \( u = \sin x \) (so that \( du = \cos x \, dx \)): \( f(u^4/(1 - u^2)^4) \, du \). As mentioned in class, this calls for partial fractions, which we won’t try to do here. (And no, there won’t be one of these on the exam).

(b) \[
\int \cos 2x \sin^4 x \, dx = \frac{1}{4} \int \cos 2x(1 - \cos 2x)^2 \, dx = \frac{1}{4} \int (\cos 2x - 2 \cos^2 2x + \cos^3 2x) \, dx
= \frac{1}{4} \left( \frac{1}{2} \sin 2x - \int (1 + \cos 4x) \, dx + \int (1 - \sin^2 2x) \cos 2x \, dx \right)
\]
[In the last integral: \( u = \sin 2x, \, du = 2 \cos 2x \, dx \)]
\[
= \frac{1}{8} \sin 2x - \frac{1}{4} x - \frac{1}{16} \sin 4x + \frac{1}{8} (\sin 2x - \frac{1}{3} \sin^3 2x) + C .
\]

(c) Let \( u = \sin x \), so that \( du = \cos x \, dx \). Then
\[
\int \frac{\cos x}{1 + \sin^2 x} \, dx = \int \frac{1}{1 + u^2} \, du = \arctan u + C = \arctan(\sin x) + C .
\]