Deriving the Taylor polynomial (in one variable)

Start with $y = f(x)$ and a point $x = a$. We want a polynomial

$$P(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + C_3(x - a)^3$$

for which

$$P(a) = f(a), \quad P'(a) = f'(a), \quad P''(a) = f''(a), \quad P'''(a) = f'''(a);$$

i.e., $P$ is the best approximation to $f$ near $a$ by a polynomial of degree at most 3. What must the coefficients $C_0$, $C_1$, $C_2$ and $C_3$ be?

$$P(x) = C_0 + C_1(x - a) + C_2(x - a)^2 + C_3(x - a)^3 \quad P(a) = C_0$$

$$P'(x) = C_1 + 2C_2(x - a) + 3C_3(x - a)^2 \quad P'(a) = C_1$$

$$P''(x) = 2C_2 + 6C_3(x - a) \quad P''(a) = 2C_2$$

$$P'''(x) = 6C_3 \quad P'''(a) = 6C_3$$

So we must have $C_0 = f(a)$, $C_1 = f'(a)$, $C_2 = \frac{1}{2}f''(a)$, and $C_3 = \frac{1}{6}f'''(a)$ to get what we want; i.e.,

$$P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \frac{1}{6}f'''(a)(x - a)^3$$

In general, if we want a polynomial of degree $N$ that has the same first $N + 1$ derivatives (including the 0-th derivative, i.e., $f$ itself) as $f$ at $a$, it would be

$$\sum_{n=0}^{N} \frac{1}{n!}f^{(n)}(a)(x - a)^n.$$