2. Let us begin by noting that \( x^2 + 2y^2 = 44 \) is closed and bounded (an ellipse), so we know that both maxima and minima of \( f \) exist on it. To find them, we need to solve the system of equations

\[
3 = \lambda(2x) \quad \text{and} \quad -2 = \lambda(4y) \quad \text{and} \quad x^2 + 2y^2 = 44.
\]

Solving for \( x \) in the first and \( y \) in the second, and substituting into the third, we get \((3/(2\lambda))^2 + 2(-2/(4\lambda))^2 = 44\), or \(1 = 16\lambda^2\), so \(x = \pm 3/(2/4) = \pm 6\) and \(y = \mp 2/(4/4) = \mp 2\), so the critical points are \((6, -2)\) and \((-6, 2)\). Now \(f(6, -2) = 22\) and \(f(-6, 2) = -22\), so \((6, -2)\) is a maximum and \((-6, 2)\) is a minimum.

3. Note first that \(x^2 - y^2 = 1\) is a hyperbola; it isn’t bounded, so there may fail to be either or both a maximum or a minimum value of \(f\) on it. In fact, as \(x\) and \(y\) both increase on one branch (the one asymptotic to \(y = x\) for positive \(x\)-values), we see that \(f\) gets arbitrarily large; so we will not find a global maximum for \(f\) on this curve. But everywhere on the curve we have \(f = (1 + y^2) + y\), which is parabola opening upward; as \(y\) gets large (positive or negative) on all the branches of the hyperbola, \(f\) increases without bound, so there will be a minimum point for \(f\) somewhere on it.

To find it, we need to solve the system

\[
2x = \lambda(2x) \quad \text{and} \quad 1 = \lambda(-2y) \quad \text{and} \quad x^2 - y^2 = 1.
\]

From the first equation, either \(x = 0\) or \(\lambda = 1\). From the third equation, \(x = 0\) gives \(-y^2 = 1\), so there are no solutions. So \(\lambda = 1\), and from the second equation \(y = -1\), so from the third equation \(x = \pm \sqrt{1 + (1/2)^2} = \pm \sqrt{5}/2\). The points \((-\sqrt{5}/2, \pm \sqrt{5}/2)\) both make \(f = \frac{3}{4}\), so they are both the desired minima.

10. The region \(x^2 + y^2 + z^2 = 1\) is closed and bounded (a sphere) so \(f\) does have maximum and minimum values on it. To find them, we need to solve the system

\[
2x = \lambda(2x) \quad \text{and} \quad -2 = \lambda(2y) \quad \text{and} \quad 4z = \lambda(2z) \quad \text{and} \quad x^2 + y^2 + z^2 = 1.
\]

From the first equation, either \(\lambda = 1\) or \(x = 0\); and from the third either \(\lambda = 2\) or \(z = 0\). From the second equation \(y = -1/\lambda\). So substituting into the fourth equation gives one of the following:

\[
\begin{array}{ccc}
\lambda &=& 1, \quad y = -1, \quad z = 0 \\
\lambda &=& 2, \quad y = -\frac{1}{2}, \quad x = 0 \\
x^2 + (-1)^2 + 0^2 &=& 1 \\
x &=& 0 \\

z &=& \pm \sqrt{3}/2 \\
y &=& 1 \quad \text{(not } -1: \lambda \neq 1) \\
\end{array}
\]

So the critical points are \((0, \pm 1, 0)\) and \((0, -\frac{1}{2}, \pm \sqrt{3}/2)\). We have \(f(0, -1, 0) = 2\), \(f(0, 1, 0) = -2\) and \(f(0, -\frac{1}{2}, \pm \sqrt{3}/2) = \frac{5}{2}\), so \((0, -\frac{1}{2}, \pm \sqrt{3}/2)\) are maxima and \((0, 1, 0)\) is a minimum.

14. The region on which we are to extremize \(f\) is shaded:

It is unbounded, so \(f\) may not attain maximum and/or minimum values on it. In fact, if we fix \(y = 0\)
18. We want to minimize \( f \) which it may do in the region; the positive \( x \)-axis is in it, we see that \( f \) can get arbitrarily large, so \( f \) has no maximum. Similarly, if we let \( y \) get large (positive) and take \( x = \sqrt{y} \) (along one edge of the region), then \( f = y - y^2 \) (in the \( yf \)-plane, a parabola opening downward), which can take on arbitrarily large negative values; so \( f \) also has no global minimum on the region. So all we might find are local extrema.

To do so, first we consider unconstrained critical points: \( 2x = 0 \) and \( -2y = 0 \) when \( x = y = 0 \), on the edge of the legal region anyway. So we look for critical points of \( f \) subject to \( x^2 - y = 0 \): Solve the system

\[
2x = \lambda (2x), \quad -2y = \lambda (2y), \quad x^2 = y
\]

to get either \( x = 0 \) (and then \( y = 0 \) also) or \( \lambda = 1 \) (and then \( y = 1/2 \) and \( x = \pm 1/\sqrt{2} \)). We have \( f(0,0) = 0 \) and \( f(\pm 1/\sqrt{2}, 1/2) = 1/4 \), so \( (0,0) \) is a local minimum on \( x^2 = y \) and \( (\pm 1/\sqrt{2}, 1/2) \) are local maxima on that curve. Are they at least local extrema for the whole region \( x^2 \geq y \)? I.e., for example, if we move into the region \( x^2 > y \) a small distance from \((0,0)\), does \( f \) increase from 0? The answer is no: If we fix \( x = 0 \) and take a small negative \( y \) (which is in the region), then \( f < 0 \). And if we move into that region away from \( (\pm 1/\sqrt{2}, 1/2) \), does \( f \) decrease? This answer is no: If we fix \( y = 1/2 \) but take \( x \) slightly larger than \( 1/\sqrt{2} \) or slightly smaller than \( -1/\sqrt{2} \), then \( f \) is larger than \( 1/4 \). So the points we found are not even local extrema on the region.

17. The region \( x^2 + y^2 \leq 1 \) is closed and bounded, so \( f \) will have maximum and minimum values on it somewhere. First we find the unconstrained critical points: \( 3x^2 = 0 \) and \( -2y = 0 \) when \( x = y = 0 \), and that is a point properly inside the legal region. [Note that \( D = (6x)(-2) - 0^2 = -12x \) is 0 at \((0,0)\), so the second derivative test doesn’t help classify this extremum. It turns out to be a saddle point, but we know there are global extrema to be found, so we’ll treat it as a candidate and test it with the others.] Turning to the edge of the region, we extremize \( f \) subject to \( x^2 + y^2 = 1 \): Solve the system

\[
3x^2 = \lambda (2x), \quad -2y = \lambda (2y), \quad x^2 + y^2 = 1 .
\]

From the first equation, either \( x = 0 \) or \( \lambda = \frac{3}{2} \); and from the second either \( y = 0 \) or \( \lambda = -1 \). So substituting into the third equation gives one of the following:

- \( x = 0 \)
- \( 0^2 + y^2 = 1 \)
- \( y = \pm 1 \)
- \( x = \pm \frac{3}{2} \)
- \( x = \pm \frac{3}{2} \)
- \( y = \pm \frac{3}{2} \)

Now \( f(0,0) = 0 \), \( f(0, \pm 1) = -1 \), \( f(1,0) = 1 \), \( f(-1,0) = -1 \) and \( f(-2/3, \pm \sqrt{5}/3) = -23/27 \) so \((1,0)\) is a (global) maximum and \((-1,0)\) and \((0, \pm 1)\) are (global) minima.

18. We want to minimize \( C = 20x + 10y + 5z \) subject to \( 1200 = 20x^{1/2}y^{1/4}z^{3/5} \), or more simply \( x^{1/2}y^{1/4}z^{3/5} = 60 \). So we need to solve the system

\[
20 = \lambda \left( \frac{1}{2} x^{-1/2} y^{1/4} z^{2/5} \right), \quad 10 = \lambda \left( \frac{1}{4} x^{1/2} y^{-3/4} z^{2/5} \right), \quad 5 = \lambda \left( \frac{2}{5} x^{1/2} y^{1/4} z^{-3/5} \right) , \quad x^{1/2} y^{1/4} z^{3/5} = 60 .
\]

Multiplying the first of these by \( 2x \), the second by \( 4y \) and the third by \( \frac{5}{2} z \) shows that \( 40x \), \( 40y \) and \( 25z \) are all equal to \( \lambda x^{1/2} y^{1/4} z^{2/5} \), so they are equal to each other: \( x = y = \frac{5}{16} z \). Substituting into the fourth equation gives

\[
\left( \frac{5}{16} \right)^{1/2} \left( \frac{5}{16} \right)^{1/4} z^{2/5} = 60 \Rightarrow \left( \frac{5}{16} \right)^{3/4} z^{3/5} = 60
\]

\[
z = \left( 60 \left( \frac{16}{5} \right)^{3/4} \right)^{20/23} \approx 75.1
\]
and \( x = y \approx \frac{3}{\pi^2} \) \( \approx 23.5 \).

23. Denote the radius, height and surface area of the cylinder by \( r \), \( h \) and \( A \) respectively. We want to minimize \( A = 2\pi r^2 + 2\pi rh \) subject to \( \pi r^2 h = 100 \), so we need to solve

\[
4\pi r + 2\pi h = \lambda (2\pi rh), \quad 2\pi r = \lambda (\pi r^2), \quad \pi r^2 h = 100;
\]

or more simply

\[
2r + h = \lambda rh, \quad r(2 - \lambda r) = 0, \quad \pi r^2 h = 100.
\]

From the second equation, because \( r = 0 \) is impossible (the cylinder must have a positive radius to have a volume), we see that \( \lambda r = 2 \), and substituting this into (the simpler version of) the first equation gives \( 2r + h = 2h \), or \( h = 2r \). Substituting this into the third equation gives \( 2\pi r^3 = 100 \), or \( r = (50/\pi)^{1/3} \); and then \( h = 2(50/\pi)^{1/3} \).