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These solutions include several problems that were not assigned: The original assignment was too long, but I had worked so hard typing the solutions that I couldn’t bring myself to delete them.

6. The line through (−1, 1) and (3, −2) is \( y - 1 = \frac{2}{3}(x - (-1)) \) or \( y = -\frac{3}{4}x + \frac{1}{4} = (1 - 3x)/4 \), so the desired integral can be written
\[
\int_{-1}^{3} \int_{-2}^{(1-3x)/4} f \, dy \, dx.
\]

7. The line through (1, 0) and (4, 1) is \( y - 0 = \frac{1}{3}(x - 1) \) or \( y = \frac{1}{3}x - \frac{1}{3} = (x - 1)/3 \), so the desired integral can be written
\[
\int_{1}^{4} \int_{(x-1)/3}^{2} f \, dy \, dx.
\]

8. The line through (0, 1) and (1, 3) is \( y - 1 = \frac{3}{2}(x - 0) \) or \( y = \frac{3}{2}x = (y - 1)/2 \), and the one through (1, 3) and (2, 1) is \( y - 3 = \frac{1-3}{2}(x - 1) \) or \( x = 1 - \frac{1}{2}(y - 3) = (5 - y)/2 \), so the desired integral can be written
\[
\int_{1}^{3} \int_{(5-y)/2}^{(5-y)/2} f \, dy \, dx.
\]

10. The diagrams of the regions of integration for the next several problems are drawn below. In the middle of the next computation, we find that the substitution \( u = x^2 \) is useful (so that \( du = 2x \, dx \) and as \( x \) varies from 0 to 2, \( u \) goes from 0 to 4):
\[
\int_{0}^{2} \int_{0}^{x} e^{x^2} \, dy \, dx = \int_{0}^{2} e^{x^2} \, dx = \int_{0}^{2} e^{x^2} [y]_{0}^{x} \, dx = \int_{0}^{2} e^{x^2} x \, dx
\]
\[
= \frac{1}{2} e^{x^2} \bigg|_{0}^{4} = \frac{e^{4} - 1}{2} \approx 26.8
\]

11. In the middle of the next computation, we find that integration by parts, with \( u = x \), \( dv = \sin x \, dx \), is useful:
\[
\int_{1}^{5} \int_{x}^{2x} \sin(x) \, dy \, dx = \int_{1}^{5} \sin(x) \int_{x}^{2x} \, dy \, dx = \int_{1}^{5} \sin(x) \, [y]_{x}^{2x} \, dx = \int_{1}^{5} x \sin(x) \, dx
\]
\[
= \left[-x \cos(x)\right]_{1}^{5} + \int_{1}^{5} \cos(x) \, dx = -5 \cos 5 + \cos 1 + \sin(x)_{1}^{5}
\]
\[
= -5 \cos 5 + \cos 1 + \sin 5 - \sin 1 \approx -2.68
\]

12.
\[
\int_{1}^{4} \int_{\sqrt{y}}^{y^2} x^{2} \, dy \, dx = \int_{1}^{4} y^{3/2} \int_{\sqrt{y}}^{y^2} \, dx \, dy = \frac{1}{3} \int_{1}^{4} (y^{6} - y^{9/2}) \, dy = \frac{1}{3} \left[ \frac{1}{7} y^{7} - \frac{2}{11} y^{11/2} \right]_{1}^{4}
\]
\[
= \frac{1}{3} \left( \frac{1}{7} (4^{7} - 1) - \frac{2}{11} (2^{11} - 1) \right) = \frac{151,555}{231} \approx 656.1
\]

13.
\[
\int_{-2}^{0} \int_{-\sqrt{9-x^2}}^{0} 2xy \, dy \, dx = \int_{-2}^{0} x[y]_{-\sqrt{9-x^2}}^{0} \, dx = \int_{-2}^{0} x(0 - (9 - x^2)) \, dx = \int_{-2}^{0} (x^3 - 9x) \, dx
\]
\[
= \left[ \frac{1}{4} x^{4} - \frac{9}{2} x^{2} \right]_{-2}^{0} = 0 - \left( \frac{(-2)^{4}}{4} - \frac{9}{2} (-2)^{2} \right) = -(4 - 18) = 14
\]
14. (a) See below.

(b) Rewriting \( x = -(y - 4)/2 \) gives \( y = 4 - 2x \), so the desired integral is

\[
\int_{0}^{2} \int_{0}^{4-2x} g(x) \, dy \, dx .
\]

15.

\[
\int_{0}^{1} \int_{y}^{1} e^{x^2} \, dx \, dy = \int_{0}^{1} \int_{0}^{x} e^{x^2} \, dy \, dx = \int_{0}^{1} xe^{x^2} \, dx = \frac{1}{2} e^{x^2} \bigg|_{0}^{1} = \frac{e}{2} - \frac{1}{2} .
\]

16.

\[
\int_{0}^{3} \int_{y^2}^{0} y \sin(x^2) \, dx \, dy = \int_{0}^{9} \int_{0}^{\sqrt{y}} y \sin(x^2) \, dx \, dy = \int_{0}^{9} \sin(x^2) \, dy \, dx = \int_{0}^{9} y \sin(x^2) \, dx
\]

\[
= \int_{0}^{9} \sin(x^2) \left[ \frac{1}{2} y^2 \right]_{0}^{\sqrt{y}} \, dx = \frac{1}{2} \int_{0}^{9} x \sin(x^2) \, dx
\]

\[
= -\frac{1}{4} \cos(x^2) \bigg|_{0}^{9} = \frac{1}{4} (1 - \cos 81)
\]
19. The surface cuts the $xy$-plane (i.e., $z = 0$) in the circle $0 = 25 - x^2 - y^2$, or $x^2 + y^2 = 25$, so we want to integrate over the interior of that circle:

$$
\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} (25 - x^2 - y^2) \, dy \, dx.
$$

20. The surface cuts the plane $z = 16$ in the circle $25 - x^2 - y^2 = 16$, or $x^2 + y^2 = 9$, so the region of integration is the interior of that circle:

$$
\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (25 - x^2 - y^2 - 16) \, dy \, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9 - x^2 - y^2) \, dy \, dx.
$$

21.
The two vertical planes meet the \( xy \)-plane (i.e., \( z = 0 \)) in the \( x \)-axis and the line \( y = x + 4 \) respectively, and the slanted plane meets the \( xy \)-plane in the line \( 2x + y = 4 \). So the region of integration is the interior of the triangle formed by those three lines. The other two lines meet the \( x \)-axis (i.e., \( y = 0 \)) at \((-4, 0)\) and \((2, 0)\); and they meet each other at \((0, 4)\). For a fixed \( y \)-value in this triangle, by solving the other two lines for \( x \), we see that the \( x \)-values run from \( y - 4 \) to \((4 - y)/2\). So the desired integral is

\[
\int_0^4 \int_{y-4}^{(4-y)/2} (4 - 2x - y) \, dx \, dy.
\]