2. 
\[ \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{\pi} \sin x \cos (y + z) \, dz \, dy \, dx = \int_{0}^{\pi} \int_{0}^{\pi} [\sin x \sin(y + z)]_{z=0}^{z=\pi} \, dy \, dx \]
\[ = \int_{0}^{\pi} \int_{0}^{\pi} \sin x (\sin(y + \pi) - \sin y) \, dy \, dx = \int_{0}^{\pi} \int_{0}^{\pi} \sin x (-\sin y - \sin y) \, dy \, dx \]
\[ = -2 \int_{0}^{\pi} \int_{0}^{\pi} \sin x \sin y \, dy \, dx = -2 \left( \int_{0}^{\pi} \sin x \, dx \right) \left( \int_{0}^{\pi} \sin y \, dy \right) \]
\[ = -2 \left( [-\cos x]_{0}^{\pi} \right)^{2} = -2(-1 - 1)^{2} = -8 \]

7. Does not make sense: The upper limit on the (middle) \( y \)-integral is \( z \), which is the variable of integration in the innermost integral.

9. Does not make sense: The upper limit on the (middle) \( x \)-integral involves an \( x \).

12. Note that this problem is very similar to Chapter 15, Section 2, problem 21; the difference is that the base is in the plane \( z = -6 \), which also changes the limits on the variables. Here is the graph:

The slanted plane \( 2x + y + z = 4 \) meets the plane \( z = -6 \) in the line \( 2x + y = 10 \); so these two planes and the plane \( y - x = 4 \) meet at \((2, 6, -6)\) (by solving \( 2x + y = 10 \) and \( y - x = 4 \) simultaneously). Also the
lines $2x + y = 10$ and $y - x = 4$ in the plane $z = -6$ meet the plane $y = 0$ at $(5,0,-6)$ and $(-4,0,-6)$ respectively. So:

$$\text{Volume} = \int_0^6 \int_{y-4}^{\frac{10-y}{2}} \int_{-6}^{\frac{4-2x-y}{2}} 1 \, dy \, dx \, dz = \int_0^6 \int_{y-4}^{\frac{10-y}{2}} (10-2x-y) \, dx \, dy \, dz$$

$$= \int_0^6 \left[ (10-y)x - \frac{x^3}{3} \right]_{y-4}^{\frac{10-y}{2}} \, dy$$

$$= \int_0^6 \left( \frac{1}{2}(10-y) - \frac{y^3}{3} \right) \, dy = \int_0^6 \left( 9 - \frac{3}{2}y^2 \right) \, dy = \int_0^6 \left( 81 - 27y + \frac{9}{4}y^2 \right) \, dy$$

$$= \left[ 81y - \frac{27}{2}y^2 + \frac{3}{4}y^3 \right]_0 = 162$$

13. The slanted plane, which can be written $z = 6 - 2x - 3y$, meets the $x$-axis at $(3,0,0)$ and the $xy$-plane in $(x/3) + (y/2) = 1$, i.e., $y = 2(1 - x/3)$. So:

$$\text{Mass} = \int_0^3 \int_0^{2(1-x/3)} \int_0^{6-2x-3y} (x+y) \, dx \, dy \, dz = \int_0^3 \int_0^{2(1-x/3)} (x+y)(6-2x-3y) \, dy \, dx$$

$$= \int_0^3 \int_0^{2(1-x/3)} \left( 6x + 6y - 2x^2 - 5xy - 3y^2 \right) \, dy \, dx$$

$$= \int_0^3 \left[ (6x - 2x^2)y + (3 - \frac{5}{2}x)y^2 - y^3 \right]_0^{2(1-x/3)} \, dx$$

$$= \int_0^3 \left( (6x - 2x^2)(2 - \frac{2}{3}x) + (3 - \frac{5}{2}x)(2 - \frac{2}{3}x)^2 - (2 - \frac{2}{3}x)^3 \right) \, dx$$

$$= \int_0^3 \left( 12x - 4x^2 - 4x^2 + \frac{4}{3}x^3 + 12 - 8x + \frac{4}{3}x^2 - 10x + \frac{20}{3}x^2 - \frac{10}{9}x^3 \right) \, dx$$

$$- 8 + 3x - \frac{8}{3}x^2 + \frac{8}{27}x^3 \, dx$$

$$= \int_0^3 \left( 4 + 2x - \frac{8}{3}x^2 + \frac{14}{27}x^3 \right) \, dx = \left[ 4x + x^2 - \frac{8}{9}x^3 + \frac{7}{54}x^4 \right]_0 = \frac{15}{2}$$

14. The volume of the region in question is $2^3 = 8$, so:

$$\text{Average} = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 (x^2 + y^2 + z^2) \, dx \, dy \, dz = \frac{1}{8} \int_0^2 \int_0^2 \left[ (x^2 + y^2)z + \frac{1}{3}z^3 \right]_0^2 \, dy \, dx$$

$$= \frac{1}{8} \int_0^2 \int_0^2 \left( 2x^2 + 2y^2 + \frac{8}{3} \right) \, dy \, dx = \frac{1}{8} \int_0^2 \left[ 2x^2y + \frac{2}{3}y^3 + \frac{8}{3}y \right]_0^2 \, dx$$

$$= \frac{1}{8} \int_0^2 \left( 4x^2 + \frac{32}{3} \right) \, dx = \frac{1}{8} \left[ 4 \frac{1}{3}x^3 + \frac{32}{3} \right]_0^2 = \frac{1}{8} \left( \frac{32}{3} + \frac{64}{3} \right) = 4$$

16. First we need the mass of the solid. (I evaluated this integral, but it’s fine if you just wrote it down and called it $M$ — or whatever — in the other formulas.)

$$\text{Mass} = M = \int_0^1 \int_0^1 \int_0^{x+y+1} 1 \, dx \, dy \, dz = \int_0^1 \int_0^{x+y+1} (x+y+1) \, dy \, dx$$

$$= \int_0^1 \left[ xy + \frac{1}{2}y^2 + y \right]_0^1 \, dx = \int_0^1 (x + \frac{3}{2}) \, dx = \left[ \frac{1}{2}x^2 + \frac{3}{2}x \right]_0^1 = 2.$$
So the desired coordinates are:

\[
\begin{align*}
\bar{x} &= \frac{1}{2} \int_0^1 \int_0^1 \int_0^{x+y+1} x \, dz \, dy \, dx \\
\bar{y} &= \frac{1}{2} \int_0^1 \int_0^1 \int_0^{x+y+1} y \, dz \, dy \, dx \\
\bar{z} &= \frac{1}{2} \int_0^1 \int_0^1 \int_0^{x+y+1} z \, dz \, dy \, dx
\end{align*}
\]

Though you weren’t asked to evaluate them, these coordinates turn out to be \(\left(\frac{13}{24},\frac{13}{24},\frac{25}{24}\right)\).