1. (Total 10 pts.) Consider the network in Figure 1.

![Figure 1: Network N.](image)

(a) (3 pts.) Determine all values of $a$ for which $N$ contains the edge labelled $a$ in one of its minimal spanning trees.

$$a \leq 10.$$ 

(b) (3 pts.) Determine all values of $a$ for which $N$ contains a unique minimal spanning tree.

$$a \neq 10.$$ 

(c) (4 pts.) For $a = 4$ list the edges of a minimal spanning tree. You may identify them in Figure 2.

$$1, 2, 2, 3, 4, 4, 7, 9, 11$$

2. (Total 10 pts.)

Let $M$ be the network from Figure 3.

(a) (2 pts.) For $a = 7$ determine the shortest paths in $M$ from the leftmost vertex $p$ to any other vertex.
Figure 2: Mark the edges of a minimal spanning tree for $a = 4$ in $N$.

Figure 3: Network $M$.

The corresponding tree contains the following edges:

$(p, q), (p, r), (q, s), (q, t), (s, u), (s, v), (u, w), (u, x), (x, y)$.

(b) (2 pts.) Is the shortest path tree obtained in (a) unique? Explain!

YES, there are no ties in the run of Dijkstra’s algorithm.

(c) (3 pts.) Determine all values of a positive integer $a$ for which the edge labelled $a$ lies on a shortest path of $M$ from leftmost vertex $p$ to the rightmost vertex $y$.

$a = 1$.

(d) (3 pts.) Determine all values of the positive integer $a$ for which the edge labelled $a$ lies on some shortest path of $M$.

$a \leq 18$. 
3. (Total 10 pts.) We form "words" (sequences of letters with repetition) composed of the 26 letters of the alphabet.

(a) (2 pts.) How many eight-letter "words" are there?

\[ 26^8 = 208827064576 \]

(b) (2 pts.) How many eight-letter words are there with exactly one of the five vowels (but that vowel may be repeated)?

\[ 5 \times (22^8 - 21^8) \]

(c) (2 pts.) How many eight-letter words are there that are the same when the order of their letters is inverted and no vowel appears more than twice (e.g. abbiibba).

\[
21^4 + C(5, 1)P(4, 1) \times 21^3 + C(5, 2)P(4, 2) \times 21^2 + C(5, 3)P(4, 3) \times 21 + C(5, 4)P(4, 4) \\
= 21^4 + 20 \times 21^3 + 120 \times 21^2 + 240 \times 21 + 120 = 437781
\]

(d) (4 pts.) Repeat all three subproblems for \( n \)-letter words with letters taken from an alphabet having \( v \) vowels and \( k \) consonants.

\[
(m + k)^n \\
\times v(m + 1)^n - k^n \\
m = \lceil n/2 \rceil, \sum_{m=k}^{n} C(v, m)P(m, 1) \times k^{m-1} + C(v, 2)P(m, 2) \times k^{m-2} + \ldots
\]

4. (Total 10 pts.) How many ways are there to pick 3 different cards from a standard 52-card deck such that:

(a) (3 pts.) The first card is not a King, the second card is a Queen, and the third card is an Ace?

\[ 50 \times 4 \times 4 - 4 \times 4 \times 4 = 46 \times 4 \times 4 = 736 \]

(b) (3 pts.) The first card is a spade, the second card is a diamond, and the third card is neither a spade nor a diamond?

\[ 13 \times 13 \times 26 = 4394 \]
(c) (4 pts.) The first card is a spade, the second card is an Ace, and the third card is not a diamond?

\[
38 + 2 \times 37 + 12 \times 38 + 12 \times 3 \times 37 = 1900
\]

5. (Total 10 pts.) We are forming 10-digit sequences with digits 1,2,3,4,5,6,7,8.

(a) (2 pts.) How many sequences can be formed?

\[8^{10}.
\]

(b) (2 pts.) How many sequences can be formed with exactly two 2s, three 3s, and four 4s?

\[C(10, 2)C(8, 3)C(5, 4) \times (8 - 3) = 63000
\]

(c) (3 pts.) How many sequences can be formed with exactly four different digits?

Let \(p_k\) denote the number of 10-digit sequences with exactly \(k\) digits and let \(q_k\) denote the number of 10-digit sequences with at most \(k\) digits. Clearly \(q_k = k^{10}\). Also \(p_1 = q_1 = 1\).

\[
p_2 = q_2 - 2p_1 = 2^{10} - 2 = 1022
\]

\[
p_3 = q_3 - C(3, 2)p_2 - C(3, 1)p_1 = 3^{10} - 32^{10} + 3 = 55980
\]

\[
p_4 = q_4 - C(4, 3)p_3 - C(4, 2)p_2 - C(4, 1)p_1 = 818520.
\]

The answer is \(C(8, 4)p_4\).

(d) (3 pts.) How many nondecreasing sequences can be formed?

\[C(17, 7) = 19448
\]

6. (Total 10 pts.) In a bridge deal, what is the probability that:

(a) (2 pts.) West has five spades, three hearts, three diamonds, and two clubs?

The total number of deals is given by \(T = 52!/(13!)^4 = 536...440000\).

Let \(F = C(13, 5)C(13, 3)C(13, 3)C(13, 2)(52-13)!/(13!)^3 = 693...8400\).

The probability is \(F/T = 0.0129307\).
(b) (2 pts.) North and South have five spades, West has three spades, and East has no spade?

Let
\[ S = C(13, 5)C(52-13, 8)C(8, 5)C(52-13-8, 8)C(3, 3)C(52-13-16, 10). \]
The probability is \( S/T = 0.000746002 \).

(c) (3 pts.) One player has all the Aces and all the Kings?

Let \( A = 4C(52-8, 13-8)(52-13)!/(13!)^3 \).
The probability is \( A/T = 22/3215975 = 6.84085 \times 10^{-6} \).

(d) (3 pts.) East has no face cards (J,Q,K, or A)?

Let \( Q = C(52-16, 13)(52-13)!/(13!)^3 \).
The probability is \( Q/T = 0.00363896 \).

7. (Total 10 pts.) Evaluate the following sums, where \( n \) is a positive integer.

(a) (1 pt.) \( C(n, 0) + C(n, 1) + C(n, 2) + C(n, 3) + ... = 2^n \)

(b) (3 pts.) \( C(n, 0) + C(n, 2) + C(n, 4) + C(n, 6) + ... = 2^{n-1} \)

(c) (3 pts.) \( C(n, 1) + C(n, 3) + C(n, 5) + C(n, 7) + ... = 2^{n-1} \)

(d) (3 pts.) \( C(n, 0) - 2C(n, 1) + C(n, 2) - 2C(n, 3) + ... = -2^{n-1} \)

8. (Total 10 pts.) How many integer solutions are there to
\[ x_1 + x_2 + x_3 + x_4 = 2003 \]
with:

(a) (3 pts.) \( x_i \geq 0 \).

\( C(4 + 2003 - 1, 4 - 1) = 1343358020 \)

(b) (3 pts.) \( x_i > 0 \).

\( C(4 + 2003 - 4 - 1, 4 - 1) = 1335334000 \)

(c) (4 pts.) Write the generating function for \( a_r \) the number of integer solutions to \( x_1 + x_2 + x_3 + x_4 = r \) with
\[ x_1 \geq 0, x_2 \geq 3, x_3 \geq 5, x_4 \geq x_2. \]

\[ x^{11}/[(1 - x)^4(1 + x)] = x^{11} + 3x^{12} + 7x^{13} + ... \]
9. (Total 10 pts.) A partition is self-conjugate if the Ferrers diagram of the partition is equal to its own transpose.

(a) (2 pts.) List all partitions of the integers 6 and 7.

6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1,
2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2,
3 + 2 + 1 + 1, 3 + 1 + 1 + 1 + 1, 2 + 2 + 2 + 1, 2 + 2 + 1 + 1 + 1,
2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1

(b) (2 pts.) Write the first six terms of the generating function for \( a_r \) the number of partitions of \( r \).

\[ 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \ldots \]

(c) (2 pts.) List all self-conjugate partitions of integers \( r = 1, 2, 3, 4, 5, 6, 7 \).

1, 2 + 1, 2 + 2, 3 + 1 + 1, 3 + 2 + 1, 4 + 1 + 1 + 1

(d) (4 pts.) Write the generating function for \( b_r \), the number of partitions of \( r \) into distinct odd integers.

\[ (1 + x)(1 + x^3)(1 + x^5)\ldots = 1 + x + x^3 + x^4 + x^5 + x^6 + x^7 + \ldots \]

10. (Total 10 pts.) Let \( L_r \) denote the mirror L-shaped network composed of \( 2r - 1 \) blocks, see Figure 4 for the case \( r = 3 \). The lower left corner is denoted \( A \) and the upper right corner is denoted \( B \). Let \( a_r \) denote the number of walks (shortest paths) from \( A \) to \( B \) in \( L_r \).

(a) (2 pts.) Determine \( a_3 \), the number of shortest paths from corner \( A \) to corner \( B \) in \( L_3 \).

\[ a_3 = 10 \]

(b) (2 pts.) Determine the values of \( a_r \), for \( r = 0, 1, 2, 3, 4, 5 \).

\[ a_0 = 1, a_2 = 2, a_3 = 10, a_4 = 17, a_5 = 26 \]
(c) (3 pts.) Determine the formula for \( a_r \).

\[ a_r = r^2 + 1 \]

(d) (4 pts.) Write the generating function for \( a_r \).

\[ \frac{(2x^2 - x + 1)/(1-x^3)}{1-x} \]

11. (Total 10 pts.) Let \( a_r = r^2 \) and let \( s_n = 1^2 + 2^2 + \ldots + n^2 \).

(a) (3 pts.) Find a generating function for \( a_r \).

\[ \frac{(x^2 + x)/(1-x^3)}{1-x} \]

(b) (3 pts.) Find a generating function for \( s_n \).

\[ \frac{(x^2 + x)/(1-x^4)}{1-x} \]

(c) (4 pts.) Evaluate the sum \( s_n = 1^2 + 2^2 + \ldots + n^2 \).

\[ s_n = n(n+1)(2n+1)/6. \]