1. (20 points)

(a) Suppose that $g$ is continuous at a $t$-value $a$; that there is an interval around $a$ (say $V_\beta(a)$ where $\beta > 0$) for which, for all $t$ in the interval except $t = a$, $g(t) \neq g(a)$; and that $\lim_{x\to g(a)} f(x) = L$. Prove that $\lim_{t\to a}(f\circ g)(t) = L$. 

(b) (This part shows why one of the hypotheses in (a) is needed, and also why we must take some care in proving the Chain Rule.) Let 

$$f(x) = \begin{cases} 
0 & \text{if } x \neq 2 \\
1 & \text{if } x = 2
\end{cases} \quad \text{and} \quad g(t) = \begin{cases} 
t \sin \frac{1}{t} + 2 & \text{if } t \neq 0 \\
2 & \text{if } t = 0
\end{cases}.$$ 

Prove that $\lim_{x\to 2} f(x) = 0$ and $g$ is continuous at $t = 0$, but that $\lim_{t\to 0}(f\circ g)(t)$ doesn’t exist. (Hint: Consider two sequences of $t$-values converging to 0, $(1/n)$ and $(1/(n\pi))$.)

2. (20 points) In this problem, you may assume without proof that sin and cos are continuous functions on all of $\mathbb{R}$ and that $\lim_{t\to 0}((\sin t)/t) = 1$.

(a) Use the identity $\sin A - \sin B = 2 \sin \left(\frac{1}{2}(A - B)\right) \cos \left(\frac{1}{2}(A + B)\right)$ to find the derivative of sin at the $x$-value $c$, from the limit definition of derivative. (In this process, of course, you are proving that sin is differentiable at $c$).

(b) Assume that cos is also differentiable, and use the derivative rules and the identity $\sin^2 A + \cos^2 A = 1$ to find the derivative of cos.

3. (30 points) Let $I$ be an interval and $f : I \to \mathbb{R}$ be differentiable on all of $I$. Assume that $f'(x) \neq 0$ for all $x \in I$.

(a) Prove that either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$.

(b) For the remaining parts of this problem, assume $f'(x) > 0$ for all $x \in I$. Prove that $f$ is strictly increasing on $I$.

(c) By the Intermediate Value Theorem, $f(I)$ is an interval $J$. Then by (b), $f$ is a one-to-one onto function $I \to J$. Thus, there is an inverse function $f^{-1} : J \to I$ defined by, for $y \in I$, $f^{-1}(y)$ is the unique $x \in I$ for which $f(x) = y$. Prove that $f^{-1}$ is also strictly increasing.

(d) Prove that $f^{-1}$ is continuous at each $y$-value $d \in J$. (So as not to make many cases, assume $d$ is an interior point of $J$, so that $f^{-1}(d)$ is an interior point of $I$.)

Remark: In a way similar to but messier than (d), it is possible to prove that, for each $d \in J$, $(f^{-1})'(d)$ exists and equals $1/f'(f^{-1}(d))$.

3.5. (8 points) However we define the function exp on $\mathbb{R}$ (which is really $e^x$ in a different notation, but we don’t want to other properties of that function), we should be able to verify from that definition that $\exp(0) = 1$ and $\exp$ is (differentiable and) equal to its own derivative for all $x \in \mathbb{R}$. Assume that the inverse function ln of exp is also differentiable, and use the fact that $\exp'(x) = \exp(x)$ to find the derivative of $\ln(y)$ (with respect to $y$). (I have switched to the variable $y$ because I think of the domain of $\exp$ as a subset of the $x$-axis and its range, the domain of $\ln$, as a subset of the $y$-axis. Hint: What is $\exp(\ln(y))$?)
4. (22 points) Recall that a function $f$ from a subset $A$ of $\mathbb{R}$ into $\mathbb{R}$ is called contractive iff there is a constant $s \in (0, 1)$ such that, for all $x_1, x_2 \in A$, we have $|f(x_1) - f(x_2)| \leq s|x_1 - x_2|$. 

(a) Prove that a contractive function is uniformly continuous.

(b) Prove that, if $A$ is a compact interval $[a, b]$, $f'(x)$ is continuous on $[a, b]$ and $-1 < f'(x) < 1$ for all $x \in [a, b]$, then $f$ is contractive.

(c) Give an example to show that, if the interval $[a, b]$ is replaced by the interval $[a, \infty)$ (still closed but no longer bounded) in (b), then the conclusion fails.
Math 323 — Solutions to Exam III

1. (a) Let \( \varepsilon > 0 \) be given, and take \( \delta > 0 \) such that, if \( 0 < |x - g(a)| < \delta \), then \( |f(x) - L| < \varepsilon \). Then pick \( \gamma > 0 \) such that \( \gamma < \beta \) and, if \( |t - a| < \gamma \), then \( |g(t) - g(a)| < \delta \). Then for each \( t \) for which \( 0 < |t - a| < \gamma \), we have \( 0 < |g(t) - g(a)| < \delta \) and hence \( |f(g(t)) - L| < \varepsilon \). Therefore, \( \lim_{t \to a} (f \circ g)(t) = L \).

(b) To see that \( g \) is continuous at \( t = 0 \), we only need to note that
\[
\lim_{t \to 0} g(t) = \lim_{t \to 0} (t \sin \frac{1}{t} + 2) = 0 + 2 = 2 = g(0).
\]
And of course \( \lim_{x \to 0} f(x) = \lim_{x \to 2} 0 = 0 \). For the sequence \((1/n)\) of \( t \)-values converging to 0, we have \( g(1/n) \neq 2 \) for all \( n \in \mathbb{N} \) (because \( \sin \) is never 0 at a natural number \( n \)), so that \( f(g(1/n)) = 0 \) and hence \( \lim (f \circ g)(1/n) = 0 \). But the sequence of \( t \)-values \((1/(n\pi))\) also converges to 0 and \( g(1/(n\pi)) = 2 \) for all \( n \in \mathbb{N} \), so \( f(g(1/(n\pi))) = 1 \), so \( \lim (f \circ g)(1/(n\pi)) = 1 \). Because we can find sequences of \( t \)-values converging to 0 for which the resulting sequences of \((f \circ g)(t)\)-values have different limits, \( \lim_{t \to 0}(f \circ g)(t) \) does not exist.

2. (a) Using the definition of derivative, the given identity, Problem 1(a), the Algebraic Limit Theorem and the continuity of \( \cos \), we have:
\[
\sin'(c) = \lim_{x \to c} \frac{\sin x - \sin c}{x - c} = \lim_{x \to c} \frac{2 \sin(\frac{1}{2}(x - c)) \cos(\frac{1}{2}(x + c))}{x - c} = \lim_{x \to c} \left[ \sin(\frac{1}{2}(x - c)) \cdot \cos(\frac{1}{2}(x + c)) \right] = 1 \cdot \cos(\frac{1}{2}(c + c)) = \cos c.
\]
(b) Because the function \( \sin^2 x + \cos^2 x \) is constant, its derivative is 0; but by the derivative rules and (a), its derivative is \( 2 \sin x \cos x + 2 \cos x \cos' x \). Setting this equal to 0 and solving for \( \cos' x \) gives \( \cos' x = -\sin x \) (at least at all \( x \)-values where \( \cos x \neq 0 \)).

3. (a) Assume BWOC that \( \exists x_1, x_2 \in I \) for which \( f'(x_1) < 0 \) and \( f'(x_2) > 0 \). Then by the Intermediate Value Property for derivatives (Darboux’s Theorem), there is an \( x_3 \) between \( x_1 \) and \( x_2 \), and hence also in \( I \), for which \( f'(x_3) = 0 \). This contradiction of the hypothesis proves the result.

(b) Let \( x_1, x_2 \in I \) with \( x_2 > x_1 \). Then by the MVT, there is an \( x_3 \) between \( x_1, x_2 \) for which \( f(x_2) - f(x_1) = f'(x_3)(x_2 - x_1) \). By hypothesis, \( f'(x_3) > 0 \), and \( x_2 - x_1 > 0 \) also, so their product is (strictly) positive. Thus, \( f(x_2) - f(x_1) > 0 \), i.e., \( f(x_2) > f(x_1) \).

(c) Assume BWOC that \( \exists y_1, y_2 \in J \) with \( y_2 > y_1 \) but \( f^{-1}(y_2) \leq f^{-1}(y_1) \). Then because \( f \) is strictly increasing, we have \( y_2 = f(f^{-1}(y_2)) \leq f(f^{-1}(y_1)) = y_1 \), a contradiction.

(d) (Because we are proving the continuity of a function from \( y \)-values to \( x \)-values, I will reverse the roles of \( \delta \) and \( \varepsilon \).) Let \( \delta > 0 \) be given, and pick \( a, b \in I \) with \( f^{-1}(d) - \delta \leq a < f^{-1}(d) < b < f^{-1}(d) + \delta \). Set \( \varepsilon = \min\{|d - f(a), f(b) - d|\} \); then for \( y \in J \) with \( |y - d| < \varepsilon \), we have \( f(a) < y < f(b) \), so \( a < f^{-1}(y) < b \), so \( |f^{-1}(y) - f^{-1}(d)| < \delta \). So \( f^{-1} \) is continuous at \( d \).

3.5. Because \( \exp(\ln(y)) = y \), we have:
\[
\ln'(y) = \frac{1}{\exp'(\ln(y))} = \frac{1}{\exp(\ln(y))} = \frac{1}{y}.
\]
4. (a) Let $\varepsilon > 0$ be given. Then for all $x_1, x_2 \in A$ such that $|x_1 - x_2| < \varepsilon$, $|f(x_1) - f(x_2)| \leq s|x_1 - x_2| < |x_1 - x_2| < \varepsilon$.

(b) Because $f'$ is continuous on the closed interval $[a, b]$, there are $x$-values $c, d \in [a, b]$ for which $f'(c) \leq f'(x) \leq f'(d)$ for all $x \in [a, b]$, so that $|f(x)| \leq \max\{|f(c)|, |f(d)|\} = s < 1$ for all $x \in [a, b]$. Now, for $x_1, x_2 \in [a, b]$, by the MVT there is an $x_3$ between them for which $f(x_1) - f(x_2) = f'(x_3)(x_1 - x_2)$, and so $|f(x_1) - f(x_2)| = |f'(x_3)||x_1 - x_2| \leq s|x_1 - x_2|$. Therefore, $f$ is contractive on $[a, b]$.

(c) Let $f$ be a function whose derivative approaches 1 from below as $x \to \infty$: Say $f'(x) = x^2/(x^2+1) = 1-1/(x^2+1)$, so $f(x) = x - \arctan(x)$. Then because $f'$ is not bounded away from 1, there is no $s \in (0, 1)$ for which $|f(x_1) - f(x_2)|/|x_1 - x_2| \leq s$: If we take $x_1 = x$ and $x_2 = x - 1$, then as $x \to \infty$, $|f(x_1) - f(x_2)|/|x_1 - x_2| = 1 - (\arctan(x) - (\arctan(x - 1))$ approaches 1.