Let $x = \sqrt{2}$. Then $x^2 = 2$. Hence, $x^2 - 2 = 0$.

a) Let $\epsilon > 0$. Let $\delta = \epsilon$. Then, for all $x$ such that $|x - \sqrt{2}| < \delta$, we have $|x^2 - 2| < \epsilon$. This shows $f(x)$ is continuous at $x = \sqrt{2}$.

b) Let $x = -\sqrt{2}$. Then $x^2 = 2$. Hence, $x^2 + 2 = 4$. But $\sqrt{2}$ is irrational, so $f(x)$ is discontinuous at $x = -\sqrt{2}$.

c) Let $x = 1$. Then $x^2 = 1$. Hence, $f(1) = 1$. But $\lim_{x \to 1} f(x) = \lim_{x \to 1} x^2 = 1$. So $f(x)$ is continuous at $x = 1$.

d) Let $x = 0$. Then $x^2 = 0$. Hence, $f(x) = 0$. But $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$. So $f(x)$ is continuous at $x = 0$.

e) Let $x = \sqrt{2}$. Then $x^2 = 2$. Hence, $f(x) = \sqrt{2}$. But $\lim_{x \to \sqrt{2}} f(x) = \lim_{x \to \sqrt{2}} x^2 = 2$. So $f(x)$ is discontinuous at $x = \sqrt{2}$.

f) Let $x = -\sqrt{2}$. Then $x^2 = 2$. Hence, $f(x) = -\sqrt{2}$. But $\lim_{x \to -\sqrt{2}} f(x) = \lim_{x \to -\sqrt{2}} x^2 = 2$. So $f(x)$ is discontinuous at $x = -\sqrt{2}$.

g) Let $x = 0$. Then $x^2 = 0$. Hence, $f(x) = 0$. But $\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 = 0$. So $f(x)$ is continuous at $x = 0$.

h) Let $x = \sqrt{3}$. Then $x^2 = 3$. Hence, $f(x) = \sqrt{3}$. But $\lim_{x \to \sqrt{3}} f(x) = \lim_{x \to \sqrt{3}} x^2 = 3$. So $f(x)$ is continuous at $x = \sqrt{3}$.

i) Let $x = -\sqrt{3}$. Then $x^2 = 3$. Hence, $f(x) = -\sqrt{3}$. But $\lim_{x \to -\sqrt{3}} f(x) = \lim_{x \to -\sqrt{3}} x^2 = 3$. So $f(x)$ is discontinuous at $x = -\sqrt{3}$.

For all other values of $x$, $f(x)$ is continuous.
\(|s_n| < M\) for all \(n\). Let \(s_n = \frac{p_n}{q_n}\). Choose \(s_{n_2} = \frac{p_2}{q_2}\) so that \(q_2 > q_1\). Since \((s_n)\) is bounded, if we assume no such \(q_2\) exists then all rationals in \((s_n)\) are of the form \(\frac{p}{q}\) where \(b \leq p_1\) and \(|a| < Mb\). But this is only a finite number of rational and we are in the case where \((s_n)\) contains infinitely many rationals. Hence, \(q_2\) exists. Continue to choose \(s_n = \frac{p_n}{q_n}\) where \(q_1 < q_2 < q_3 < \cdots\). Hence, \(f(s_n) = \frac{1}{q_n} \to 0 = f(\bar{x})\).

17.15 (\(\iff\)) The only difference between this and the definition is that \(x_0\) is not allowed to be in the sequence. Since the definition states that \((x_n) \to x_0 \Rightarrow f(x_n) \to f(x_0)\) for all sequences \((x_n)\) in the domain, it is true, in particular, for those sequences that do not contain \(x_0\).

(\(\Rightarrow\)) If \(f\) is continuous at \(x_0\), then for every \((x_n) \to x_0\) we have \(f(x_n) \to f(x_0)\). Thus, for every \((x_n) \to x_0\) with \(x_n \neq x_0\) for any \(n\), we still have \(f(x_n) \to f(x_0)\).

18.2) The subsequential limit is not necessarily in \((a, b)\).

18.3) \(f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)\). Setting this \(= 0\) gives us \(x = 1, 3\) as possible relative maxima. We have \(f''(x) = 6x - 12\) so that \(f''(1) < 0\) while \(f''(3) > 0\). Thus, \(x = 1\) is a relative maximum and \(x = 3\) is a relative minimum. We must check this against any endpoints. We have only one endpoint: \(x = 0\). We find \(f(0) = 1\), \(f(1) = 5\), and \(f(3) = 1\). Now consider \(\lim_{x \to 5} f(x) = 21\). Hence, \(x = 0, 3\) are both minimum, and \(f\) has no maximum on \([0, 5]\).

18.4) Consider \(f(x) = \frac{1}{x - x_0}\) defined for \(x \in S\). Since, \(x_0 \notin S\), we see that \(f\) is continuous on \(S\) but is clearly unbounded.

18.5) a) Consider \(h(x) = f(x) - g(x)\), for \(x \in [a, b]\), a continuous function on \([a, b]\). Note that \(h(a) \geq 0\) and \(h(b) \leq 0\). If \(h(a) = 0\) or \(h(b) = 0\) then \(f(a) = g(a)\) or \(f(b) = g(b)\) and we are done. Hence, we may assume that \(h(a) > 0\) and \(h(b) < 0\). Since \(h(b) < 0 < h(a)\), by IVT there exists \(x_0\) such that \(h(x_0) = 0\). Hence, \(f(x_0) = g(x_0)\).

b) Take part (a) with \(a = 0, b = 1\), and \(g(x) = x\).

18.6) Note that \(f(x) = \cos x\) maps \((0, \pi/2)\) into \((0, 1) \subseteq (0, \pi/2)\). Extend \(f\) to \([0, \pi/2]\). We have \(f(0) = 1\) and \(f(\pi/2) = 0\) and \(f : [0, \pi/2] \to [0, \pi/2]\). By 18.5 with \(f(x) = \cos x\) and \(g(x) = x\), we have \(x_0 \in [0, \pi/2]\) such that 

\[\cos x_0 = x_0.\]

Since we know \(x_0 \neq \pi/2\), we have \(x_0 \in (0, \pi/2)\) and are done.

18.7) Write the equation as \(x = 2^{-x}\) and extend the domain to \([0, 1]\). Clearly, \(x \neq 0, 1\). By 18.5, with \(g(x) = x\) and \(f(x) = 2^{-x}\), we have \(x_0 \in [0, 1]\) such that \(x_0 = 2^{-x_0}\), i.e., \(x_0 2^{x_0} = 1\). Since \(x_0 \neq 0, 1\), we are done.

18.8) If \(f(a) f(b) < 0\) then we have \(f(a) > 0\) and \(f(b) < 0\) or \(f(a) < 0\) and \(f(b) > 0\). Either way, \(0\) is between \(f(a)\) and \(f(b)\). By IVT, there exists \(x_0\) such that \(f(x_0) = 0\) with \(x_0 \in (a, b)\).

18.9) Note that \(\lim_{x \to \infty} f(x) = -\infty\) while \(\lim_{x \to -\infty} f(x) = \infty\). Given \(M > 0\) and \(m < 0\), we can find \(x_1, x_2\) such that \(f(x_1) < m\) and \(f(x_2) > M\). Hence, \(0\) is between \(f(x_1)\) and \(f(x_2)\), so by IVT there exists \(x_0 \in (x_1, x_2)\) such that \(f(x_0) = 0\), i.e., \(f\) has at least one real root.

18.12) a) This was already done.

b) Let, without loss of generality, \(f(a) < y < f(b)\) with \(a < b\). We must prove that there exists \(x_0\) such that \(f(x_0) = y\). First, if \(a > 0\), then there is nothing to be done since \(f\) is continuous on \([a, \infty)\). If \(b < 0\), there is also nothing to do here. Hence, the only cases to consider are when \(a \leq 0 \leq b\) (where at least one inequality is strict). If \(a = 0\), we are considering \(0 < y < f(b) \leq 1\). Since \(\sin x = y\) always has a solution (infinitely many), say \(\sin c = y\), take \(x_0 = \frac{1}{c+2\pi n}\), where \(n\) is large enough so that \(x_0 \in (0, b)\). If \(b = 0\), the argument is similar. So, consider \(a < 0 < b\). If \(f(a) f(b) < 0\), then either \(f(a) < y < 0 < y < f(b)\), or \(y = 0\). The first two have been dealt with, and if \(y = 0\) we have \(f(0) = 0\). If \(f(a) f(b) = 0\), this has also been taken care of. So, \(f(a) f(b) > 0\) remains. We will assume \(f(a) > 0\) (where \(f(a) < 0\) is similar). We have \(0 < f(a) < y < f(b) < 1\). Clearly \(\sin(x) = y\) has a solution, but we need \(x \in (a, b)\). Let \(\sin c = y\) and consider \(x_0 = \frac{1}{c+2\pi n}\). We claim that for some \(n\), we have \(x_0 \in (a, b)\). Assume not, i.e., assume \(\frac{1}{c+2\pi n} > b\) while \(\frac{1}{c+2\pi(n+1)} < a\). Then \(a > 0\), which contradicts \(a\) being negative. This completes the proof.