Coorbit spaces and discretizations

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Bergman spaces

Let $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$ and define the Bergman spaces

$$A^{p,\alpha} = \{ f \in \mathcal{O}(D) \mid \int_D |f(z)|^p (1 - |z|^2)^{\alpha - 2} \, dz < \infty \}$$

We will now look at a different characterization of some of these Bergman spaces using square integrable representations.

We will be interested only in the case of a non-integrable representation, since the other case has already been covered by Feichtinger and Gröchenig.

We will also see some discretization results for these spaces.
The affine group is $\mathbb{R}_+ \times \mathbb{R}$ can be regarded as the subgroup

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} a + a^{-1} + ib & b + i(a - a^{-1}) \\ b - i(a - a^{-1}) & a + a^{-1} - ib \end{pmatrix} \right\} \subseteq SU(1, 1)$$

for $a > 0$ and $b \in \mathbb{R}$.

$G$ can be represented on the Hilbert space $H = A^{2,2}$ by

$$\pi \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = \left(-\beta z + \bar{\alpha}\right)^{-2} f\left(\frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}}\right)$$
Bergman spaces

The smooth vectors for this representation have been characterized by [Olafsson, Ørsted]

\[ H^\infty_\pi = \left\{ \sum_k a_k z^k \mid \forall m : \sum_k \left| a_k \right|^2 \frac{2^{2m} k^{4m}}{k + 1} < \infty \right\} \]

with (conjugate) dual

\[ H^{-\infty}_\pi = \left\{ \sum_k a_k z^k \mid \exists m : \sum_k \left| a_k \right|^2 \frac{2^{-2m} k^{-4m}}{k + 1} < \infty \right\} \]

Denote the conjugate dual pairing by \( \langle \cdot , \cdot \rangle \) and define

\[ V_1(f)(a, b) = \langle f , \pi(a, b)1 \rangle. \]
Bergman spaces

It can be shown that

\[ A^p,p = \{ f \in \mathcal{O}(D) \mid \int_D |f(z)|^p (1 - |z|^2)^{p-2} \, dz < \infty \} \]

can also be described by

\[ A^p,p = \{ f \in H_{\pi}^{-\infty} \mid V(f) \in L^p \} \]

with equivalent norm \( \| f \| = ||V(f)||_{L^p} \) for \( p > 1 \).

**Note:** \( V_1(1) \not\in L^1 \), but \( L^p \ast |V_1(1)| \subseteq L^p \).
Notation

- Let $S$ be a Frechét space which is weakly dense in its conjugate dual $S^*$ and let $\langle v', u \rangle$ denote the conjugate dual pairing of $u \in S$ and $v' \in S^*$.
- Let $\pi$ be a representation of a (Lie) group on $S$ and define the voice transform $V_u(v')(x) = \langle v', \pi(x)u \rangle$.
- Denote by $\pi$ also the contragradient representation on $S^*$, i.e. $\langle \pi(x)v', u \rangle = \langle v', \pi(x^{-1})u \rangle$.
- Let $Y$ be some left- and right-invariant (quasi) Banach Function space.
The generalization (Coorbit spaces)

Assume there is a cyclic $u \in S$ such that

R1. $V_u(v) \ast V_u(u) = V_u(v)$ for all $v \in S$

R2. $Y \ast V_u(u) \subseteq Y$ and $f \mapsto f \ast V_u(u)$ is continuous

R3. If $f = f \ast V_u(u) \in Y$ then $v \mapsto \int f(x) \langle \pi(x)u, v \rangle \, dx$ is in $S^*$

R4. $S^* \ni v' \mapsto \int \langle \pi(x)u, u \rangle \langle v', \pi(x)u \rangle \, dx$ is continuous

and then define

$$\text{Co}_S^u Y = \{ v' \in S^* | V_u(v') \in Y \}$$

with norm $\| v' \| = \| V_u(v') \|_Y$

**Theorem**

- $\text{Co}_S^u Y$ is a $\pi$-invariant Banach space
- $V_u : \text{Co}_S^u Y \rightarrow Y \ast V_u(u) \subseteq Y$ is an isometric isomorphism
Feichtinger/Gröchenig

Let \((\pi, H)\) be a unitary irreducible square integrable representation of \(G\), and \(w\) a submultiplicative weight on \(G\). Assume that \(Y\) is such that

\[
Y \ast L^1_w(G) \subseteq Y \quad \text{and} \quad \| F \ast f \|_Y \leq C \| F \|_Y \| f \|_{1,w}
\]

Further assume there is a \(u \neq 0\) s.t.

\[
H^1_w = \{ v \in H | V_u(v) \in L^1_w(G) \} \ni u
\]

Then \(H^1_w\) is a Banach space with norm \(\| v \| = \| V_u(v) \|_{1,w}\), and (R1-R4) are satisfied with \(S = H^1_w\). The coorbit space is then

\[
Co_{FG} Y = \{ v \in (H^1_w)^* | V_u(v) \in Y \}
\]

with norm \(\| v \|_{FG} = \| V_u(v) \|_Y\).
Sampling

Sampling can be done on \( Y \ast V_u(u) \) instead of on \( \text{Co}Y \).

- \( \{x_i\} \) is a countable subset of \( G \)
- \( U \) is a compact neighbourhood of the identity for which \( \# \{j \mid x_i U \cap x_j U \neq \emptyset\} < N \) for all \( i \).
- \( \{\psi_i\} \) is a partition of unity \( \text{supp} \psi_i \subseteq x_i U \)

**Theorem** ([Gröchenig]) If it is possible to pick \( u \) and \( U \) such that

\[
V_u(u)^\# = \sup_{x \in U^{-1}} |r_x V_u(u) - V_u(u)| \in L^1_w
\]

and \( \|V_u(u)^\#\|_{L^1_w} < 1/\|V_u(u)\|_{L^1_w} \) then

\[
Sf = \sum_i f(x_i) \psi_i \ast V_u(u)
\]

is continuous with continuous inverse on \( Y \ast V_u(u) \).
Sampling in Bergman spaces

For the Bergman spaces it is still possible to show that $S$ is invertible even without the integrability condition.

**Lemma**
We can pick $U$ such that $V_1(1)^\# \leq C_U |V_1(1)|$ and as $U \to \{e\}$ we have $C_U \to 0$.

Using this lemma it can be shown that we can pick $U$ such that the operator

$$Sf = \sum_i f(x_i) \psi_i \ast V_1(1)$$

is continuous with continuous inverse on $L^p \ast V_1(1)$.
Sampling in Bergman spaces

Similar results can be shown for

$$Tf = \sum_{i} c_{i} f(x_{i}) \ell_{x_{i}} V_{1}(1)$$

where $$c_{i} = \int \psi_{i}(x) dx$$.

The idea for the proof of this was found in [Zhu].
Further projects

1. Coorbit spaces for homogeneous spaces and nilpotent Lie groups.

2. Bandlimited functions are also coorbits. The kernel involved (sinc function) is again not $L^1$. Sampling theorems exist, but we cannot prove estimates like in the lemma before. We would like discretization machinery to handle general coorbits.
References I


References II


- Zhu, Kehe, Spaces of holomorphic functions in the unit ball, Graduate Texts in Mathematics, 226, Springer-Verlag, New York, 2005