3.8 Exponential Growth and Decay

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Population growth

Example

If \( y = f(t) \) is the number of individuals in a population of animals or humans at time \( t \), then it seems reasonable to expect that the rate of growth \( f'(t) \) is proportional to the population.

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where $k$ is a constant.
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- If $k > 0$ the equation is called the **law of natural growth**.
- If $k < 0$ the equation is called the **law of natural decay**.
- It is an example of a **differential equation**.
The only solution of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}.$$
Example

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where \( P \) is the population?
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- Use the model to estimate the population in 1993 and to predict the population in 2020.
Population growth

\[ P = 2560e^{0.01785t} \]

Population (in millions)

Years since 1950
If \( m(t) \) is the mass remaining from an initial mass \( m_0 \) if the substance after time \( t \), then the relative decay rate

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- \frac{1}{m} \frac{dm}{dt} = -k,
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where \( k \) is a negative constant.
Radioactive decay

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The **half-life** is the time required for half of any given quantity to decay.
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- Find the mass after 1,000 years correct to the nearest milligram.
The half-life of radium-226 is 1590 years.

- A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after $t$ years.
- Find the mass after 1,000 years correct to the nearest milligram.
- When will the mass be reduced to 20 mg?
Newton’s Law of Cooling

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**Newton’s Law of Cooling** is the following differential equations

\[
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\]

Set \( y(t) = T(t) - T_s \); then the equation becomes

\[
\frac{dy}{dt} = ky.
\]
Example

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- What is the temperature of the soda pop after another half hour?
- How long does it take for the soda pop to cool to 50°F?
Example

A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?
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2. How much of the sample remains after 100 years?
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2. How much of the sample remains after 100 years?
3. After how long will only 1 mg remain?