Summary of the Method of Undetermined Coefficients

The Method of Undetermined Coefficients is a method for finding a particular solution to the second order nonhomogeneous differential equation

\[ my'' + by' + ky = g(t) \]

when \( g(t) \) has a special form, involving only polynomials, exponentials, sines and cosines.

In the following table, \( P_n(t) \) is a polynomial of degree \( n \):

\[ P_n(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0. \]

<table>
<thead>
<tr>
<th>( g(t) )</th>
<th>( y_p(t) ) (first guess)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ke^{rt} )</td>
<td>( Ae^{rt} )</td>
</tr>
<tr>
<td>( k \cos(\omega t) \text{ or } k \sin(\omega t) )</td>
<td>( A \sin(\omega t) + B \cos(\omega t) )</td>
</tr>
<tr>
<td>( P_n(t) )</td>
<td>( A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0 )</td>
</tr>
<tr>
<td>( P_n(t)e^{rt} )</td>
<td>( (A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) e^{rt} )</td>
</tr>
<tr>
<td>( P_n(t)e^{rt} \cos(\omega t) \text{ or } P_n(t)e^{rt} \sin(\omega t) )</td>
<td>( (A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0) e^{rt} \cos(\omega t) + (B_n t^n + B_{n-1} t^{n-1} + \cdots + B_1 t + B_0) e^{rt} \sin(\omega t) )</td>
</tr>
</tbody>
</table>

If any term in the first guess is also a solution to the corresponding homogeneous equation, multiply the whole guess by \( t \). If any term in this second guess is still a solution to the homogeneous equation, multiply by \( t \) again (i.e. multiply the first guess by \( t^2 \)).

To find the coefficients, substitute \( y_p \) into the differential equation, and collect the coefficients of the different functions of \( t \).

**Example.** Consider \( y'' + 3y' + 2y = t^2 \). We find \( y_h(t) = k_1 e^{-t} + k_2 e^{-2t} \). Since \( g(t) = t^2 \), a second degree polynomial, we use the third line in the above table, and we guess \( y_p(t) = A_2 t^2 + A_1 t + A_0 \). None of the three terms in this guess also solves the homogeneous equations, so this guess will work.

**Example.** Consider \( y'' + 6y' + 10 = te^{-3t} \cos(t) \). We find \( y_h(t) = k_1 e^{-3t} \cos(t) + k_2 e^{-3t} \sin(t) \). Now \( g(t) = te^{-3t} \cos(t) \), so we use the fifth line in the table above \( (n = 1, a_1 = 1, a_0 = 0, r = -3, \omega = 1) \) to make the first guess

\[
y_p(t) = (A_1 t + A_0) e^{-3t} \cos(t) + (B_1 t + B_0) e^{-3t} \sin(t)
\]

\[
= A_1 t e^{-3t} \cos(t) + A_0 e^{-3t} \cos(t) + B_1 t e^{-3t} \sin(t) + B_0 e^{-3t} \sin(t).
\]

However, the terms \( A_0 e^{-3t} \cos(t) \) and \( B_0 e^{-3t} \sin(t) \) both solve the homogeneous equation, so we must multiply the first guess by \( t \). Our guess is now

\[
y_p(t) = t \left\{ (A_1 t + A_0) e^{-3t} \cos(t) + (B_1 t + B_0) e^{-3t} \sin(t) \right\}
\]

\[
= A_1 t^2 e^{-3t} \cos(t) + A_0 t e^{-3t} \cos(t) + B_1 t^2 e^{-3t} \sin(t) + B_0 t e^{-3t} \sin(t).
\]

None of the terms in this guess solves the homogeneous equation, so this guess will work.