3.5.4

\[ \cosh x \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \]  

(a) To differentiate this sine series, we must use equation (3.4.13), with \( f(x) = \cosh x \):

\[ \sinh x \sim \frac{1}{L}(\cosh(L) - 1) + \sum_{n=1}^{\infty} \left[ \frac{n\pi}{L} b_n + \frac{2}{L}((-1)^n \cosh(L) - 1) \right] \cos(n\pi x/L) \]  

This is a cosine series, so to differentiate again, we can simply differentiate term-by-term:

\[ \cosh x \sim \sum_{n=1}^{\infty} \left( \frac{n\pi}{L} \right) \left[ \frac{n\pi}{L} b_n + \frac{2}{L}((-1)^n \cosh(L) - 1) \right] \sin(n\pi x/L) \]  

Now equate the coefficients in (1) and (3):

\[ b_n = \left( \frac{n\pi}{L} \right) \left[ \frac{n\pi}{L} b_n + \frac{2}{L}((-1)^n \cosh(L) - 1) \right] \]  

Then solve for \( b_n \):

\[ b_n = \frac{2n\pi(1 - (-1)^n \cosh(L))}{L^2 + (n\pi)^2} \]  

(You can verify this answer by using the formula for the Fourier sine series coefficients; you’ll have to integrate by parts twice.)

(b) We use definite integrals from 0 to \( x \). In particular, \( \int_0^x \cosh t \, dt = \sinh x - \sinh 0 = \sinh x \), and \( \int_0^x \sinh t \, dt = \cosh x - \cosh 0 = \cosh x - 1. \)

Integrating (1) results in

\[ \sinh x = \sum_{n=1}^{\infty} -\frac{L}{n\pi} b_n \cos(n\pi x/L) - 1 \]  

\[ = \sum_{n=1}^{\infty} \frac{L}{n\pi} b_n + \sum_{n=1}^{\infty} \left( -\frac{L}{n\pi} b_n \right) \cos(n\pi x/L) \]  

The first infinite series on the right is the constant term of the Fourier cosine series for \( \sinh x \), so we can replace it with \( (1/L) \int_0^L \sinh(x) \, dx = (\cosh(L) - 1)/L \):

\[ \sinh x = \frac{1}{L}(\cosh(L) - 1) + \sum_{n=1}^{\infty} \left[ -\frac{L}{n\pi} b_n \right] \cos(n\pi x/L) \]
Integrate again from 0 to $x$:

$$\cosh(x) - 1 = \frac{1}{L}(\cosh(L) - 1)x + \sum_{n=1}^{\infty} \left[ -\left( \frac{L}{n\pi} \right)^2 b_n \right] \sin(n\pi x/L)$$  \hspace{1cm} (8)

or

$$\cosh(x) = 1 + \frac{1}{L}(\cosh(L) - 1)x + \sum_{n=1}^{\infty} \left[ -\left( \frac{L}{n\pi} \right)^2 b_n \right] \sin(n\pi x/L)$$  \hspace{1cm} (9)

We want the right side to be a Fourier sine series, so there can not be a constant term, nor a term with a constant multiplied by $x$. To fix this, we use the Fourier sine series for the constant 1 and for $x$:

$$1 \sim \sum_{n=1}^{\infty} \frac{2}{n\pi} (1 - (-1)^n) \sin(n\pi x/L), \quad x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi} (-1)^{n+1} \sin(n\pi x/L)$$  \hspace{1cm} (10)

After substituting these series into (9) and collecting the coefficients of $\sin(n\pi x/L)$, we find

$$\cosh(x) = \sum_{n=1}^{\infty} \left[ \frac{2}{n\pi} (1 + \cosh(L)(-1)^{n+1}) - \left( \frac{L}{n\pi} \right)^2 b_n \right] \sin(n\pi x/L)$$

$$= \sum_{n=1}^{\infty} \left[ \frac{2}{n\pi} (1 - \cosh(L)(-1)^n) - \left( \frac{L}{n\pi} \right)^2 b_n \right] \sin(n\pi x/L)$$  \hspace{1cm} (11)

By equation the coefficients in (1) and (11), and then solving for $b_n$, we obtain (5).