Canards and Horseshoes in the Forced van der Pol Equation

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Brief History

\[ x'' + d(x^2 - 1)x' + x = a \sin(\nu \tau) \]

- 1926 van der Pol \((a = 0)\): relaxation oscillations
- 1927 van der Pol and van der Mark: hysteresis and bistability
- 1940... Cartwright and Littlewood (chaos before “chaos”)
- 1949 Levinson: simplified piecewise linear model. Inspired...
- 1963 Smale: Horseshoe Map
- 1978 Levi: further simplified model, symbolic dynamics
- 1980’s Grasman (et al): Asymptotic analysis
- ... Many more analytical and numerical studies
Rewrite in Standard Fast/Slow Form

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New parameters: \[ \varepsilon = 1/d^2, \quad \omega = \frac{\nu d}{2\pi} \]

New variables: \[ t = \sqrt{\varepsilon \tau}, \quad \theta = \omega t, \quad y = \varepsilon \dot{x} + x^3/3 - x \]

Then

\[ \varepsilon \dot{x} = x - \frac{1}{3}x^3 + y \]

\[ \dot{y} = -x + a \sin(2\pi \theta) \]

\[ \dot{\theta} = \omega \]
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Symmetry: \( x \rightarrow -x, \quad y \rightarrow -y, \quad \theta \rightarrow \theta + 1/2 \)
Slow and Fast Subsystems

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\[\varepsilon = 0\]

**Slow Subsystem (DAE)**

\[y = \frac{1}{3} x^3 - x\]
\[\dot{y} = -x + a \sin(2\pi \theta)\]
\[\dot{\theta} = \omega\]
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Eliminate \( y \) and desingularize
\[ \dot{\theta} = \omega (x^2 - 1) \]
\[ \dot{x} = -x + a \sin(\theta) \]
(Time reversed for \( |x| < 1 \).)
Slow and Fast Subsystems

\[
\begin{align*}
\varepsilon \dot{x} &= x - \frac{1}{3} x^3 + y \\
\dot{y} &= -x + a \sin(2\pi \theta) \\
\dot{\theta} &= \omega
\end{align*}
\]

\[
\begin{align*}
t \rightarrow \varepsilon t \\
\dot{x} &= x - \frac{1}{3} x^3 + y \\
\dot{y} &= \varepsilon (-x + a \sin(2\pi \theta)) \\
\dot{\theta} &= \varepsilon \omega
\end{align*}
\]

\[
\varepsilon = 0
\]

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\text{Eliminate } y \text{ and desingularize}
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**Slow and Fast Subsystems**

Slow Subsystem (DAE)
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\begin{align*}
\dot{x} & = x - \frac{1}{3}x^3 + y \\
\dot{y} & = -x + a \sin(2\pi \theta) \\
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\end{align*}
\]

Fast Subsystem
\[
\begin{align*}
\dot{x} & = x - \frac{1}{3}x^3 + y \\
\dot{y} & = \varepsilon (-x + a \sin(2\pi \theta)) \\
\dot{\theta} & = \varepsilon \omega
\end{align*}
\]

Eliminate \( y \) and desingularize
\[
\begin{align*}
\dot{\theta} & = \omega (x^2 - 1) \\
\dot{x} & = -x + a \sin(\theta)
\end{align*}
\]

(Time reversed for \(|x| < 1\).)
Slow and Fast Subsystems

\[ \varepsilon \ddot{x} = x - \frac{1}{3} x^3 + y \]
\[ \dot{y} = -x + a \sin(2\pi \theta) \]
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\[ t \rightarrow \varepsilon t \]

\[ \dot{x} = x - \frac{1}{3} x^3 + y \]
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\[ \varepsilon = 0 \]

\[ \text{Slow Subsystem (DAE)} \]
\[ y = \frac{1}{3} x^3 - x \]
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\[ \text{Eliminate } y \text{ and desingularize} \]

\[ \dot{\theta} = \omega (x^2 - 1) \]
\[ \dot{x} = -x + a \sin(\theta) \]
(Time reversed for \( |x| < 1. \))

\[ \text{Fast Subsystem} \]
\[ \dot{x} = x - \frac{1}{3} x^3 + y \]
\[ \dot{y} = 0, \dot{\theta} = 0 \]

\[ \varepsilon = 0 \]

\[ \text{Critical Manifold} \]
\[ y = \frac{1}{3} x^3 - x \]
Phase Space
Phase Space

Critical Manifold, Slow Subsystem
Phase Space

Fast Subsystem
Example: Periodic Orbit
Phase Space
Phase Space
Phase Space
Canards at the Folded Saddle

- If $a > 1$, there is a pair of folded equilibria (pseudo-singular points) on each fold. One is a saddle, the other is either a node or a spiral.
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- **Canards** form at folded saddles.
  (Benoit 1983; Mischenko, Kolesov, Kolesov, & Rhozov 1994; Szmolyan & Wechselberger 2003)
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- *Canards* form at folded saddles.
  (Benoit 1983; Mischenko, Kolesov, Kolesov, & Rhozov 1994; Szmolyan & Wechselberger 2003)

- Representative System:

  \[ \varepsilon \dot{x} = y + x^2 \]
  \[ \dot{y} = -ax + bx \]
  \[ \dot{z} = 1 \]

  Critical manifold is $y = -x^2$.
  The origin is a folded equilibrium.
Canards at a Folded Saddle
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Canards at a Folded Saddle

![Graph showing canards at a folded saddle](Warren Weckesser, Equadiff 2003 – p. 11)
Canards at a Folded Saddle

[Diagram showing a plot with axes x and z, and a line labeled 'Fold']
Canards at a Folded Saddle

Maximal canard
Canards at a Folded Saddle
Canards at a Folded Saddle
Canards at a Folded Saddle

![Graph showing the behavior of canards at a folded saddle point. The graph plots the system's phase portrait, highlighting the dynamics near the saddle point, with trajectories indicating the canard phenomenon.]
Canards at a Folded Saddle
Forced van der Pol Poincaré Map
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Canard forms
Forced van der Pol Poincaré Map

Σ₂ “Jump away” canard
Forced van der Pol Poincaré Map

Forced van der Pol Poincaré Map

Maximal canard

Forced van der Pol Poincaré Map

“Jump back” canard
Forced van der Pol Poincaré Map
Forced van der Pol Poincaré Map
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Forced van der Pol Poincaré Map

\[\Sigma_1\]
\[\Sigma_2\]
Forced van der Pol Poincaré Map

Symmetry: $\Sigma_2 \rightarrow \Sigma_1$
Horseshoe in the Forced van der Pol System

“Cartoon”

- Jump Across Canards
- Maximal Canard
- Jump Back Canards
- No Canards

$\kappa \epsilon$
Horseshoe in the Forced van der Pol System

“Cartoon”

Jump Across Canards
Maximal Canard

No Canards
Jump Back Canards
No Canards

Numerical Computation

$\varepsilon = 0.00010, a = 1.1000, \omega = 1.5500$
Bifurcations of Periodic Orbits

Bifurcations of Periodic Orbits

Maximal Canard

\[ L_2 \text{ Norm of } x \]

- 3Aa: \( bn \)
- 3Sa: \( nn \)
- 12a: \( bnbbnbnb \)
- 6a: \( bnb \)
- 3Sb: \( bb \)
- 6b: \( anbb \)
- 6c: \( anbn \)
- 12b: \( anb \)anan
- 12c: \( anbnabab \)
- 3Ac: \( ab \)
- 3Sc: \( aa \)
- 3Ab: \( an \)

Bifurcations of Periodic Orbits

Example from full shift on three symbols
**Example: Maximal Canard Bifurcation**

Periodic orbit at the limit point has a maximal canard.

- Maximal canard: crosses fold, follows unstable manifold all the way back to fold, then jumps.
Example: Maximal Canard Bifurcation

Periodic orbit on a branch leading to the left-most limit point (symbols \( \text{bb} \)).

\[
\begin{align*}
\text{canard: crosses fold} \\
\text{jumps at fold} \\
\text{jumps back}
\end{align*}
\]
Example: Periodic Orbits and Symbolic Dynamics

Periodic orbit with symbol sequence anbbanan.
Summary

- A detailed numerical study combined with an understanding of the fast/slow dynamics provides a clear picture of the horseshoe map in the forced van der Pol system.
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- A detailed numerical study combined with an understanding of the fast/slow dynamics provides a clear picture of the horseshoe map in the forced van der Pol system.

- Canards that form at nonhyperbolic points of the slow manifold play a key role in this picture.
References


