

Supplement: Conditional statements and basic methods of proof

Many of the facts we'll establish in the theory of numbers come to us as conditional statements. It's usually an easy matter to detect a conditional statement because they have a standard formulation:

If (some stated hypothesis: H), then (some stated conclusion: C).

Note, each of the following is a conditional statement

If a is an integer and $0 < a$, then $-a < 0$.

If a and b are integers and $ab = 0$, then $a = 0$ or $b = 0$.

If a , b , and c are integers and $a + c < b + c$, then $a < b$.

However, conditional statements can take on slightly different formulations. For example:

"For any integer a , $a(0) = 0$ " is an economical rephrasing of the more familiar "If a is taken to be any arbitrary integer, then $a(0) = 0$."

"The statement $ab = 0$ implies $a = 0$ or $b = 0$ " is a rephrasing of "If $ab = 0$, then $a = 0$ or $b = 0$."

In the first of these examples, the hypothesis has been replaced by a universal quantifying phrase, "For any" In the latter, the (transitive) verb "implies" joins the hypothesis to the conclusion in a way that makes their relationship clear.

Be sure you can spot conditional statements, their hypotheses, and their conclusions.

In order to discuss different methods of proving a conditional statement, we need to consider the truth value of such. Consider the following truth table:¹

H	C	If (H), then (C)
True	True	True
True	False	False
False	True	True
False	False	True

First note that there is only one set of circumstances under which a conditional statement is false: The hypothesis is true and the conclusion is false. Therefore to establish that a conditional statement is false, it suffices to produce a specific counterexample; that is, a specific situation for which the hypothesis is true and the conclusion is false. Here's an example:

If $ab = 0$, then $a = 0$ and $b = 0$. Clearly this is a false statement since we can offer $a = 3$ and $b = 0$ as a counterexample. (For this choice, $ab = 0$; however the conclusion is false since a isn't 0.)

Note that in order to convince someone a conditional statement is false, you don't have to argue in general terms such as: " $ab = 0$ doesn't imply $a = 0$ and $b = 0$ because a and b could be chosen so that $ab = 0$, but either a or b isn't 0." Cut to the chase; all you need to do is give a single counterexample.

The truth table above also tells us that a conditional statement is always true whenever the hypothesis fails to hold true. In such cases, we refer to the hypothesis as vacuous.

¹ I assume here that you've worked with truth tables in the past.

If $10 < 5$, then $15 < 10$. Although it may bother you, this statement is true since the hypothesis is vacuous.

(If this bothers you, think of a conditional statement as a promise: “If it rains tomorrow, then I’ll give you a dollar,” for example. If it fails to rain on the day, then I can’t break my promise regardless of whether I decide to give you a dollar or not. Either way, I’m true to my promise.)

So, in order to establish that a conditional statement is true, there’s only one situation that matters: The truth of the hypothesis must ensure the truth of the conclusion. This observation provides an outline for the direct method for proving a conditional statement: If (H), then (C).

- Assume H is true
- Deduce C must be true using H, any known axioms, other established truths, logic, and your wits.

There is another method that’s used to prove a conditional statement true; it uses the contrapositive of the original statement. The contrapositive of the statement “If (H), then (C)” is the statement “If (the opposite C), then (the opposite of H).” We sometimes write “not H” for “the opposite of H.”

Consider for example: “If $ab = 0$, then $a = 0$ or $b = 0$.” The contrapositive of the this statement is “If neither a nor b is 0, then $ab \neq 0$.”

Extending our truth table above to include the contrapositive form, we have

H	C	If (H), then (C)	not C	not H	If (not C), then (not H)
True	True	True	False	False	True
True	False	False	True	False	False
False	True	True	False	True	True
False	False	True	True	True	True

That is, the truth values of the original conditional statement and its contrapositive agree regardless of the truth values associated with either H or C. This means that the two forms of the statement are logically equivalent in structure. That is, the truth of the contrapositive form ensures the truth of the original conditional statement, and vice versa. With this we can outline the contrapositive method of proof of the statement “If (H), then (C)” as follows

- Assume the opposite of C is true
- Deduce that the opposite of H must be true.

Note this approach is nothing more than a direct method of proof applied to the (equivalent) contrapositive statement.

Finally, there is another powerful method of proof that we’ll exploit: it’s usually called a proof by contradiction. Here’s the outline of the method applied to the statement “If (H), then (C)”

- Assume H is true
- Assume the opposite of C is true
- Produce a statement that is both true and false; that is, deduce a contradiction

If you can do this, then you know C cannot be false in the presence of H – because logic (usually) abhors contradictions.

Although this method looks a bit like a contrapositive proof, it is different. Note in particular that both methods begin and end at different ‘places.’

Method	Beginning assumption(s)	Goal
Contrapositive	The opposite of C	The opposite of H
Contradiction	H and the opposite of C	Any contradiction

While a proof by contradiction gives you more to work with by way of assumptions, it is not generally clear where the proof will end. That is, the contradiction that ends the proof could come from anywhere. With a contrapositive proof, you have less to work with by way of assumptions, but you have a basic idea of where the proof should be going: establishing the truth of the opposite of H.

You'll see examples of all three of these methods in the introductory sections of the course.