Math 250: Number Theory and Mathematical Reasoning
Valente

Section 2.2: The Pythagorean relation and Fermat Conjecture

Definition 2.3 A pythagorean triple is an ordered triple of integers $\{x, y, z\}$ that satisfies the [quadratic] diophantine equation $X^2 + Y^2 = Z^2$.

The triple is called a primative pythagorean triple [ppt] if $(x, y, z) = 1$.

Lemma 1 If $\{x, y, z\}$ is a ppt, then exactly one of x or y is even and the other is odd.

NB. As a convenient convention, given any ppt $\{x, y, z\}$ we will take x to be the even integer.

Lemma 2 Let $\{x, y, z\}$ be a ppt. Then $(x, y) = 1$; $(y, z) = 1$; and $(x, z) = 1$.

NB. $(x, y, z) = 1$ does not imply that $(x, y) = 1$ in general.

Lemma 3 Let a, b, c, and n be naturals with $ab = c^n$. If $(a, b) = 1$, then there are natural numbers a' and b' for which $a = (a')^n$ and $b = (b')^n$.

Theorem 2.4 Let x, y, and z be natural numbers with x even; y odd; and $(x, y, z) = 1$. The triple $\{x, y, z\}$ is a ppt if and only if $x = 2st$; $y = s^2 - t^2$; and $z = s^2 + t^s$ for some natural numbers s and t with

(i) $s > t$;
(ii) $(s, t) = 1$; and
(iii) exactly one of s or t is even.

Lemma 4 If $a = 4q + 1$ or $a = 4q + 3$ for some integer q, then $a^2 = 4q' + 1$ for some integer q'.

Theorem 2.5 [Fermat] The diophantine equation $X^4 + Y^4 = Z^2$ has no solution in in the naturals.

NB. The proof is based on Fermats argument by 'infinite descent'.

Corollary The diophantine equation $X^4 + Y^4 = Z^4$ has no solution in the naturals.

Fact 2.6 The diophantine equation $X^4 - Y^4 = Z^2$ has no solution in the naturals.

Corollary The area of an integral pythagorean triangle can never be a perfect integral square.