Math 250: Number Theory and Mathematical Reasoning
Valente

Section 2.3: Perfect Numbers: Part I

Definition 2.7 Let n be a natural number. We say that n is perfect if it is the sum of its positive divisors less than itself.

NB. It follows immediately from this definition that n is perfect if and only if $\sigma(n) - n = n$ or $\sigma(n) = 2n$.

Theorem 2.8 Let $k > 1$. If $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ is perfect. Further, every even perfect number is of this form.

NB. This theorem completely characterizes all even perfect numbers. The first statement was essentially proved by Euclid.

Proposition 2.9 Let $a > 0$ and $k \geq 2$. If $a^k - 1$ is prime, then $a = 2$ and k is prime.

NB. This theorem says that if $2^k - 1$ is to be prime, then k must be prime. It does NOT say that if k is prime, then $2^k - 1$ is prime. That is, the converse of this proposition is not true. You can check $2^{11} - 1$ is not prime.

Proposition 2.10 Every even perfect number ends in a 6 or an 8.

NB. While we can characterize even perfect numbers, no one knows if there are an infinite number of perfect numbers. That is, no one knows if there are infinitely many primes for which $2^p - 1$ is prime.

Are there odd perfect numbers? That's another open question. But we'll consider it in more detail in Section 4.