

A Quick Introduction to Factor Analysis

From *Data Analysis: A Statistical Primer for Psychology Students* by Edward L. Wike:

... [T]here are elaborate extensions of r to multivariate data ... another case is k variables (*factor analysis*). Fortunately, for both the student and author, none of these extensions will be considered here.

From *Linear Algebra and its Applications* by Gilbert Strang:

... [T]he technique is so much needed that, even starting as an unwelcome black sheep of multivariate analysis, it has spread from psychology into biology and economics and the social sciences.

From linear algebra:

A “matrix” is a rectangular (in this discussion, usually square) array of numbers; e.g.,

$$\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 3 & 0 \\ 3 & .5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

We can add two matrices of the same dimensions, by adding corresponding entries; and we can multiply a matrix by a number, by multiplying all the entries in the matrix by that number:

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 2 \end{bmatrix} \qquad 3 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & -3 \end{bmatrix}$$

A square matrix is a “diagonal matrix” if the only nonzero entries (if any) are on the “main diagonal” (upper left to lower right):

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The “transpose” of a matrix is the (new) matrix obtained turning the rows into columns (and vice versa):

$$\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 \\ 5 & 2 \\ 7 & 3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 3 & 0 \\ 3 & .5 & 2 \\ 0 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 3 & 0 \\ 3 & .5 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

If a (square) matrix A has the property that $A^T = A$ (as this one does), then A is called “symmetric”. This just amounts to saying that, for each i and j , the entry in row i and column j is equal to the entry in row j and column i , i.e., the matrix is symmetric about its main diagonal.

Relevant example:

Suppose we have several variables, the scores on 4 exams, one in math, one in physics, one in English, and one in history — one list of 4 scores for each student in a class. Taking the exams

in pairs, we can find a correlation coefficient for each pair. We then form a matrix with the rows and columns labelled M(ath), P(hysics), E(nglish) and H(istory); the entry in the X row and Y column is the correlation coefficient between X and Y. The result is a symmetric matrix (since the correlation between X and Y is the correlation between Y and X) with 1's on the main diagonal (since the correlation between X and X is 1):

$$\begin{array}{c} M \quad P \quad E \quad H \\ \begin{array}{l} M \\ P \\ E \\ H \end{array} \begin{pmatrix} 1 & .74 & .24 & .24 \\ .74 & 1 & .24 & .24 \\ .24 & .24 & 1 & .74 \\ .24 & .24 & .74 & 1 \end{pmatrix} \end{array}$$

We can multiply two matrices, provided that the number of columns in the first matrix is equal to the number of rows in the second matrix:

$$\begin{bmatrix} 3 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3(2) + 5(3) + 7(1) & 3(1) + 5(0) + 7(-1) \\ 1(2) + 2(3) + 3(1) & 1(1) + 2(0) + 3(-1) \end{bmatrix} = \begin{bmatrix} 28 & -4 \\ 11 & -2 \end{bmatrix}$$

Let I denote a square matrix with all 1's on the main diagonal and 0's in all other positions. Then I behaves for matrix multiplication as 1 does for number multiplication:

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = A$$

So I is called the “identity matrix” (of its size). If two square matrices A and B have the property that $AB = I$, then we call B the “inverse” of A and write $B = A^{-1}$.

Matrix multiplication is not commutative: It is often the case that $AB \neq BA$, even for square matrices of the same size (so that the products are the same size as well. But there are some rules that do work:

$$(AB)C = A(BC) \qquad (AB)^T = B^T A^T \qquad (AB)^{-1} = B^{-1} A^{-1}$$

A one-column matrix is a “column vector”. If U and V are column vectors of the same length, then their “dot product” $U^T V$ has only one row and column; i.e., it is just a number:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = 1(-1) + 2(1) + 3(0) = 1$$

For a square matrix B , the entry in row i and column j of the product $B^T B$ is the dot product of the i -th and j -th columns of B . For a few matrices B , this dot product will be 1 if $i = j$ and

0 if $i \neq j$; in other words $B^T B = I$, or $B^T = B^{-1}$. Such a matrix is called “orthogonal”. One useful fact about orthogonal matrices is that they preserve dot products; i.e., if B is an orthogonal (square) matrix and U and V are column vectors, the

$$(BU)^T(BV) = U^T B^T B V = U^T I V = (U^T V) .$$

For a square matrix A , if a column vector V and a number c are such that $AV = cV$, then V is called an “eigenvector” of A , and c is the corresponding “eigenvalue”:

$$\begin{bmatrix} 1 & .74 & .24 & .24 \\ .74 & 1 & .24 & .24 \\ .24 & .24 & 1 & .74 \\ .24 & .24 & .74 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.22 \\ 2.22 \\ 2.22 \\ 2.22 \end{bmatrix} = 2.22 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & .74 & .24 & .24 \\ .74 & 1 & .24 & .24 \\ .24 & .24 & 1 & .74 \\ .24 & .24 & .74 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1.26 \\ 1.26 \\ -1.26 \\ -1.26 \end{bmatrix} = 1.26 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Big Theorem: If A is any symmetric matrix (with real number entries), then there is a matrix B having an inverse, for which $B^T A B$ is a diagonal matrix. In this case, the columns of B are all eigenvectors of A , with corresponding eigenvalues the main diagonal entries of $B^T A B$. Moreover, this B can be chosen to be orthogonal (i.e., $B^{-1} = B^T$).

Computer programs like *Mathematica* can take a symmetric matrix A and find the corresponding orthogonal matrix B and the diagonal matrix $B^T A B$ quickly; but to do it by hand is difficult and can lead to approximation errors.

Using *Mathematica* on our earlier 4-by-4 matrix, we see that

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

are also apparently eigenvectors, both with corresponding eigenvalue .26:

$$\begin{bmatrix} 1 & .74 & .24 & .24 \\ .74 & 1 & .24 & .24 \\ .24 & .24 & 1 & .74 \\ .24 & .24 & .74 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .26 \\ -.26 \\ 0 \\ 0 \end{bmatrix} = .26 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & .74 & .24 & .24 \\ .74 & 1 & .24 & .24 \\ .24 & .24 & 1 & .74 \\ .24 & .24 & .74 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ .26 \\ -.26 \end{bmatrix} = .26 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

The dot product of any two different ones of the four eigenvectors of this matrix are already 0, but to make the dot product of any one with itself equal to 1, we need to choose the correct number multiple; and *Mathematica* has given us the required number multiple:

$$B = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad B^T A B = \begin{bmatrix} 2.22 & 0 & 0 & 0 \\ 0 & 1.26 & 0 & 0 \\ 0 & 0 & .26 & 0 \\ 0 & 0 & 0 & .26 \end{bmatrix} = C$$

Notice that, if the entries on the main diagonal of a diagonal matrix are all ≥ 0 , then it is the square of another diagonal matrix:

$$C = \begin{bmatrix} 2.22 & 0 & 0 & 0 \\ 0 & 1.26 & 0 & 0 \\ 0 & 0 & .26 & 0 \\ 0 & 0 & 0 & .26 \end{bmatrix} = \begin{bmatrix} \sqrt{2.22} & 0 & 0 & 0 \\ 0 & \sqrt{1.26} & 0 & 0 \\ 0 & 0 & \sqrt{.26} & 0 \\ 0 & 0 & 0 & \sqrt{.26} \end{bmatrix} \begin{bmatrix} \sqrt{2.22} & 0 & 0 & 0 \\ 0 & \sqrt{1.26} & 0 & 0 \\ 0 & 0 & \sqrt{.26} & 0 \\ 0 & 0 & 0 & \sqrt{.26} \end{bmatrix} = S S$$

So we get:

$$A = B C B^T = B S S^T B^T = B S S^T B^T = (B S)(B S)^T = F F^T$$

where $F = B S$.

Notice, though, that this “factorization” of A is not at all unique: If Q is any orthogonal matrix, then

$$A = F F^T = F I F^T = F (Q Q^T) F^T = (F Q)(F Q)^T = (F_1)(F_1)^T$$

where $F_1 = F Q$.

Also, if we subtract any diagonal matrix from a symmetric matrix, we get another symmetric matrix, which can be factored in the same way: For any diagonal matrix D of the same size as A ,

$$A = G G^T + D$$

where G can be multiplied by any orthogonal matrix (and other changes may also be possible) without changing the result, A .

Factor analysis takes a correlation matrix (call it R), like the one relating scores on various exams above, and tries to account for as much as of the correlations off the main diagonal with as few “factors” as possible; the factors here are weighted sort-of-averages (technical term: linear combinations) of the tests, with loadings given by the columns of the G when we write

$$R = G G^T + D$$

For instance, the matrix A that we have been using as an example works perfectly for factor analysis: If we subtract the diagonal matrix $.26I$ and diagonalize the resulting matrix, since two of the main diagonal entries in $B^T AB$ are 0, the matrix $F = BS$ has two columns of all 0's, which can be ignored when we find “factors”:

$$A = FF^T + .26I \quad \text{where} \quad F = \begin{bmatrix} .7 & .5 \\ .7 & .5 \\ .7 & -.5 \\ .7 & -.5 \end{bmatrix}$$

So we can say that all of the variability off the main diagonal in A is reflected in a “general intelligence” factor that is the same in all 4 tests (as shown in the first column of F , all .7's) and a “numerical vs. literary” factor that shows the opposition the scores in math-physics (.5 and .5 in the second column of F) and English-history (-.5 and -.5 in that column).

That interpretation looks very mathematical and irrefutable, until we remember that the representation of A in this form is not unique. For instance, we also have $A = GG^T + .26I$ where

$$G = \begin{bmatrix} .6 & \sqrt{.38} & 0 \\ .6 & \sqrt{.38} & 0 \\ .4 & 0 & \sqrt{.58} \\ .4 & 0 & \sqrt{.58} \end{bmatrix}$$

This time there doesn't seem to be a “general intelligence” factor or opposition between a numerical vs. a literary intelligence; rather the math-physics pair seems to vary independently from the English-history pair. Also, we now have 3 factors to consider instead of 2.

From Strang (p. 227):

In a typical problem [from social science], the original factors may have substantial loadings on dozens of variables, and such a factor is practically impossible to interpret. It has a mathematical meaning, as a [column vector], but no useful meaning to a social scientist. Therefore he tries [inserting an orthogonal factor Q] that produces a simple structure, with large loadings in a few components and negligible loadings elsewhere. ... Thus, two experts ... could very easily produce *completely different interpretations of the same data*.