November 18, 2008

Math 102 / Core 143 CX — Exam II

Put your answers to all the problems on the numbered sheet, except for the essay (#7), which goes on the back. The blue book is for your scratch work and will NOT be graded.

1. *(15 points)* A regular icosahedron (a 20-sided solid) has its sides numbered 1 through 20.
   (a) What is the conditional probability, given that a roll of this 20-sided die is at least 6, that it is even?
   (b) What is the probability that, on 2 rolls of this die, the first is even and the second is less than 3 (i.e., 1 or 2)?
   (c) What is the probability that, on 2 rolls of this die, the first is even <u>or</u> the second is less than 3 (or both statements are true)?
   (d) What is the probability that, on 5 rolls, at least one is less than 3?
   (e) What is the probability that, on 8 rolls, at least 7 are less than 3?

2. *(12 points)* This question refers to the same icosahedral die as in Question 1. The standard deviation of the numbers from 1 to 20 is about 5.77. Using this value, find the approximate probability that the average of 100 rolls of this die will lie between 9 and 12.

3. *(15 points)* The aces, deuces, treys, and 4's through 9's are selected from a straight (= bridge = poker) deck, making a 36-card deck in which the average card value is 5. We could (I) pick 20 cards from the deck (replacing the card and shuffling between picks) or (II) pick 40 cards (under the same conditions). In which of these two scenarios, I or II, would the following be more likely?
   (a) The average value of the cards chosen is at least 4.
   (b) The average value of the cards chosen is at least 8.
   (c) The sum of the cards chosen it at least 8 more than 5 times the number of cards chosen.
   (d) The sum of the cards chosen is at least 8 times the number of cards chosen.
   (e) More than 20% of the cards chosen are 6's (exactly). (Warning: This should change your "box model" from the one you were using in (a) through (d).)

4. *(8 points)* The test for the disgusting disease lantzeria has a sensitivity of 90% (i.e., if you have it, the test will say you have it 90% of the time) and specificity of 85% (i.e., if you don't have it, the test will say you don't 85% of the time). Suppose that 4% of the population have this disease, and that you test positive for it: what is the probability that you actually have it? (Steps in the solution: 1. Percent of the population who are sick and test positive. 2. Percent who are well and still test positive. 3. Percent who test positive. 4. Percent who are sick and test positive divided by percent who test positive.)

5. *(20 points)* For each of the following data sources, should we expect the data to follow a normal distribution, a uniform distribution (all outcomes equally frequent), or neither. For the "neither" responses, give a rough sketch of the probability histogram.
   (a) results of the RAND() function in Excel
   (b) averages salaries of large samples from a large population
   (c) sums of two rolls of a dodecahedral (12-sided) die, with the numbers 1 through 12 on its faces (so that the possible outcomes are from 2 to 24)
   (d) heights in a large population

6. *(20 points)* A market survey of 400 households in a city of 200,000 households finds that the average distance from the household to the closest grocery is 5.5 miles, with a standard deviation of 2 miles.

   (a) Find a 90% confidence interval for the average distance from a household in the city to the nearest grocery.

   (b) How many households must be surveyed in order to get a 90% confidence interval only a third as wide as the one you found in (a) (assuming the new survey has the same $\overline{x}$ and s as this one.)

7. *(10 points)* Relative to the article "Math Is Hard, Barbie Said" by Sharon Begley: Does the parenthetical comment "... U.S. girls (who are more sensitive to social status than boys)" imply a "hard-wiring" of girls' brains for more social sensitivity, or is there another explanation?

Some possibly useful formulas:

$$\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} \qquad \sqrt{\frac{\text{total population} - \text{sample size}}{\text{total population} - 1}}$$

$$\sigma = (\text{larger} - \text{smaller})\sqrt{(\text{fraction with larger})(\text{fraction with smaller})} = \sqrt{\text{average of } (x-\overline{x})^2}$$

$$EV \text{ of sum (or count)} = n \cdot (AV \text{ of box}) \qquad SE \text{ of sum (or count)} = \sqrt{n} \cdot (SD \text{ of box})$$

$$EV \text{ of avg (or \%)} = AV \text{ of box} \qquad SE \text{ of avg (or \%)} = \frac{SD \text{ of box}}{\sqrt{n}}$$

Normal table (Area between $-z$ and z)

z	Area(%)	z	Area(%)	z	Area(%)	z	Area(%)	z	Area(%)
0.0	0.0	0.9	63.19	1.8	92.81	2.7	99.31	3.6	99.968
0.05	3.99	0.95	65.79	1.85	93.57	2.75	99.4	3.65	99.974
0.1	7.97	1	68.27	1.9	94.26	2.8	99.49	3.7	99.978
0.15	11.92	1.05	70.63	1.95	94.88	2.85	99.56	3.75	99.982
0.2	15.85	1.1	72.87	2	95.45	2.9	99.63	3.8	99.986
0.25	19.74	1.15	74.99	2.05	95.96	2.95	99.68	3.85	99.988
0.3	23.58	1.2	76.99	2.1	96.43	3	99.73	3.9	99.99
0.35	27.37	1.25	78.87	2.15	96.84	3.05	99.771	3.95	99.992
0.4	31.08	1.3	80.64	2.2	97.22	3.1	99.806	4	99.9937
0.45	34.73	1.35	82.3	2.25	97.56	3.15	99.837	4.05	99.9949
0.5	38.29	1.4	83.85	2.3	97.86	3.2	99.863	4.1	99.9959
0.55	41.77	1.45	85.29	2.35	98.12	3.25	99.885	4.15	99.9967
0.6	45.15	1.5	86.64	2.4	98.36	3.3	99.903	4.2	99.9973
0.65	48.43	1.55	87.89	2.45	98.57	3.35	99.919	4.25	99.9979
0.7	51.61	1.6	89.04	2.5	98.76	3.4	99.933	4.3	99.9983
0.75	54.67	1.65	90.11	2.55	98.92	3.45	99.944	4.35	99.9986
0.8	57.63	1.7	91.09	2.6	99.07	3.5	99.953	4.4	99.9989
0.85	60.47	1.75	91.99	2.65	99.2	3.55	99.961	4.45	99.9991

Name: ______________________________

Your answers to Exam II

1.(a)

(b)

(c)

(d)

(e)

2.

3.(a)

(b)

(c)

(d)

(e)

4.

5.(a)

(b)

(c)

(d)

6.

7.

Math 102 / Core 143 — Solutions to Exam II

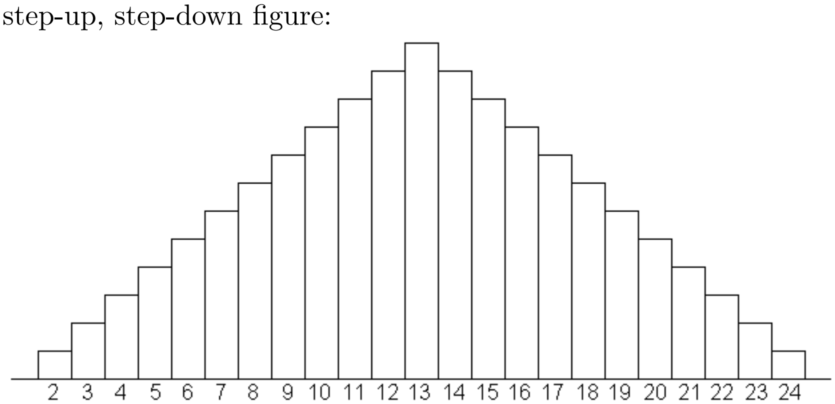
1. (a) Because 8 of the 15 numbers 6 through 20 are even, the probability is 8/15.
   (b) The probability that the first is even is 1/2, and the probability that the second is less than 3 is $2/20 = 1/10$, and the events are independent, so the probability that both happen is $(1/2)(1/10) = 1/20$.
   (c) $1/2 + 1/10 - 1/20 = 11/20$.
   (d) This is the complement of the event "getting no number less than 3 on 5 rolls", so: $1 - (9/10)^5 \approx 0.41$.
   (e) $C(8,7)(1/10)^7(9/10)^1 + C(8,8)(1/10)^8 = [8(9) + 1]/10^8 = 0.0000000073$.

2. The averages of 100 rolls have a nearly normal distribution, with an EV of 10.5 (the average of the possible values) and an SE of about $5.77/\sqrt{100} = .577$, so, using the normal table,
$$P(9 < \text{avg} < 12) = P\left(\frac{9 - 10.5}{.577} < z < \frac{12 - 10.5}{.577}\right) = P(-2.6 < z < 2.6) \approx .99\ .$$

3. (a) There is more variability in the average with smaller samples so the average is more likely to be above 8 with 20 cards: I.
   (b) Because 5 is below the expected value of the average, and the sample average tends to be closer to the expected value of the average when the samples are larger, this is more likely with II.
   (c) Because there is more variability in the sum with a larger sample, this is more likely with II.
   (d) Despite the fact that it is phrased in terms of sum, this is really the same as part (a), so the answer is I.
   (e) The "deck" is now essentially 1's and 0's, meaning 6's (of which there are 4) and non-6's (of which there are 32). With this box model, we should expect only $1/9 \approx 11\%$ of the cards to be 6's, so 20% is higher than expected; and variability in percentage is larger with small samples, so this is more likely with I.

4. Of the 96% of the population who do not have the disease, the test will erroneously say that 15%, or 14.4% of the population, actually have it. Of the 4% who have the disease, the test will say that 90% of them, or 3.6% of the population, have it. Thus, of the 18% of the population who test positive, only 3.6% actually have the disease: $3.6\%/18\% = 20\%$. So you only have about a 20% chance of actually having the disease even though you tested positive.

5. (a) Uniform.
   (b) Averages of large samples, even from a non-normal population, are normally distributed, by the Central Limit Theorem.
   (c) Neither. It is a step-up, step-down figure:



(d) Normally distributed (the cumulative effect of many small influences, usually genetic). Other answers are acceptable, for example taking into account that men's and women's average heights differ.

6. (a) From the normal table, in order to get 90% of the area, we need to go up and down 1.65 s's from $\overline{x}$: $5.5 \pm 1.65(2/\sqrt{400}) = 5.5 \pm .165$ miles, or rounding off, between 5.3 and 5.7 miles.

   (b) Because s and the z-value don't change, in order to make the interval a third as wide, we need to interview $3^2 = 9$ times as many people, or 3600.

7. You could say that hard-wiring for mathematical ability does exist, despite Begley's objections — for example, even the Math Olympiad teams that have a relatively large number of girls still have a majority of boys — and that greater sensitivity to social status is just another example of hard-wiring. Or, you could instead say that greater sensitivity to social status is, like confidence in one's mathematical abilities, enhanced or discouraged by upbringing and society. As usual, I am less interested in your position than in how you describe it, especially in giving evidence that you had read the article and thought about the question.