Lists of Axioms

More lists may be added later, if and when more "structures" are defined in the course.

Definition. A *field* is a set F with two *binary operations* $+$ and $\cdot$ (or juxtaposition) (called "addition" and "multiplication" respectively) which associate with every pair of elements a, b in F their "sum" $a + b$ and their "product" $a \cdot b$ (or ab); for which the following conditions are satisfied:

- (commutativity of addition) For all a, b in F, $a + b = b + a$.
- (associativity of addition) For all a, b, c in F, $a + (b + c) = (a + b) + c$.
- (existence of a zero) There is an element 0 of F with the property that $a + 0 = a$ for every a in F.
- (existence of negatives) For every element a of F, there is an element, denoted $-a$, of F, for which $a + (-a) = 0$.
- (commutativity of multiplication) For all a, b in F, $ab = ba$.
- (associativity of multiplication) For all a, b, c in F, $a(bc) = (ab)c$.
- (distributivity of addition over multiplication) For all a, b, c in F, $a(b+c) = (ab) + (ac)$. item (existence of a unity) There is an element 1 of F with the property that $a \cdot 1 = a$ for all a in F. (It is usual to require that $0 \neq 1$, so that every field has at least two elements.)
- (existence of reciprocals) For each element a of F <u>except</u> 0, there is an element, denoted a^{-1}, for which $a \cdot a^{-1} = 1$.

Definition. Let F be a field. A *vector space* over F is a set V with two operations on it: "addition" associates to each pair $\boldsymbol{v}, \boldsymbol{w}$ of elements of V another element of V, their "sum" $\boldsymbol{v} + \boldsymbol{w}$, and "scalar multiplication" associates to an element c of F (now called the "field of scalars") and an element $\boldsymbol{v}$ of V their "(scalar) product" $c\boldsymbol{v}$; for which the following conditions are satisfied:

- (commutativity of addition) For all $\boldsymbol{v}, \boldsymbol{w}$ in V, $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$.
- (associativity of addition) For all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in V, $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$.
- (existence of a zero) There is an element $\boldsymbol{0}$ of V with the property that $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$ for every $\boldsymbol{v}$ in V.
- (existence of negatives) For every element $\boldsymbol{v}$ of V, there is an element, denoted $-\boldsymbol{v}$, of V, for which $\boldsymbol{v} + (-\boldsymbol{a}) = \boldsymbol{0}$.
- (associativity of scalar multiplication) For every a, b in F and every $\boldsymbol{v}$ in V, $(ab)\boldsymbol{v} = a(b\boldsymbol{v})$.
- (distributivity of scalar multiplication over scalar addition) For every a, b in F and every $\boldsymbol{v}$ in V, $(a + b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$.
- (distributivity of scalar multiplication over vector addition) For every a in F and every $\boldsymbol{v}, \boldsymbol{w}$ in V, $a(\boldsymbol{v} + \boldsymbol{w}) = a\boldsymbol{v} + a\boldsymbol{w}$.
- (unity) For the unity element 1 in F and any $\boldsymbol{v}$ in V, $1\boldsymbol{v} = \boldsymbol{v}$.