Chapter 2: Solving Linear Equations

2.5. Inverse Matrices (continued)

Now let's return to the general question of finding inverse matrices. We have already seen that the inverse of an elementary matrix is another elementary matrix, and that we can write down such an inverse easily. (Well, to be fair, we've only done that for two of the three kinds of elementary matrix, but the operation of multiplying a row by a nonzero constant is easy to invert:

Example. The inverse of the elementary matrix that multiplies the third row in a 3×3 matrix by 7,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix} ,$$

is the one that multiplies the third row by 1/7,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/7 \end{bmatrix} .)$$

So it follows that we can write down the inverse of the product of elementary matrices:

$$(E_1E_2E_3)^{-1} = E_3^{-1}E_2^{-1}E_1^{-1} .$$

`At this point, switch to the presentation for Unit 3.`

2.6. Elimination = Factorization: $A = LU$

The thing to note in these examples is that the elementary row operations on the coefficient matrix A, i.e., the multiplication of A on the left by the elementary matrices gave an upper triangular matrix U. Now because elementary row operations can be "undone" by other elementary matrices, we can even write down the inverses of elementary matrices. For instance, in the first example,

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

These new elementary matrices are all lower triangular, so their product is also lower triangular:

$$\begin{aligned} E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = L . \end{aligned}$$

I've chosen to multiply them in this order because:

$$\begin{aligned} E_{32}E_{31}E_{21}A &= U \\ E_{31}E_{21}A &= E_{32}^{-1}U \\ E_{21}A &= E_{31}^{-1}E_{32}^{-1}U \\ A &= E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U = LU \ . \end{aligned}$$

Thus, A has an *"LU-decomposition"*, i.e., a factorization into a lower triangular matrix and an upper triangular matrix.

In the second example, though, the permutation matrix P_{23} is not lower triangular, and it is its own inverse, so even though we still have

$$A = E_{21}^{-1}E_{31}^{-1}P_{23}U \ ,$$

this coefficient matrix A does not have an LU-decomposition. That is why our text only resorts to reversing two equations in a system or rows in a matrix if elimination breaks down — such a reversal means that the coefficient matrix does not have an LU-decomposition. Applied mathematicians and pure mathematicians argue over the importance of the LU-decomposition: The applied mathematicians find it very useful: easy to program and <u>usually</u> exists (and we'll see other applications later); while the pure mathematicians think that something that fails to exist so often is not worth pursuing.