

## Chapter 3: Vector Spaces and Subspaces

### 3.1. Spaces of Vectors

Now we want to generalize the concept of vector space. In its most general form, we should begin with the scalars we are allowed to multiply by. They could come from any system within which you can add, subtract, multiply and (except by 0) divide, and all the usual rules of arithmetic hold. Well, maybe not “all”; we don’t need anything about order (“greater than 0”) or even the idea that adding 1’s together will never give 0 (there are fields in which that isn’t true). Some fields are: the rational numbers (integers — whole numbers — over integers), the real numbers, the complex numbers (things like  $3 + 2i$  where  $i^2 = -1$ , which you probably learned about in high school algebra), and the set that I usually refer to as  $\mathbb{Z}_2$ , but Dan Swiecki says is called  $B$  in computer science: The elements 0 and 1, with the usual rules except that  $1 + 1 = 0$ . (Remember that I said that sums of 1’s could be 0 in some fields?) So what are the rules that a field must follow? I don’t want to list them here, but I may post them on the website. Anyway, the only field we will always (or at least for a long while) use is the real numbers  $\mathbb{R}$ .

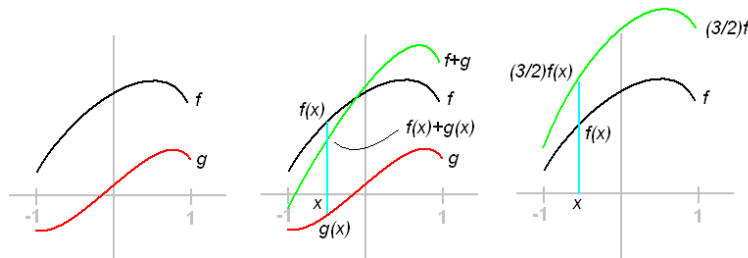
So if all the scalars are just real numbers, what are the vectors? The answer is, anything in a context in which adding two things and multiplying something by a scalar make sense, and both of these operations always give you something else in the set. Again, there are some rules that these operations must follow (these at least are listed in the book, just before the Section 3.1 problem set), but for present purposes we can just say the operations act as you would expect. Instead of dwelling much on these rules, let’s list some vector spaces (“over the reals”) that we will use frequently:

**Example.** Some real vector spaces:

- The set of real numbers is a vector space over itself: The sum of any two real numbers is a real number, and a multiple of a real number by a scalar (also real number) is another real number. And the rules work (whatever they are).
- $\mathbb{R}^n$ , for any positive integer  $n$ , is a vector space over  $\mathbb{R}$ : For example, the sum of two lists of 5 numbers is another list of 5 numbers; and a scalar multiple of a list of 5 numbers is another list of 5 numbers.
- The text uses the symbol  $\mathbf{M}$  for the set of  $2 \times 2$  matrices (with real entries — I would have preferred something more, maybe  $M_{2,2}(\mathbb{R})$ , because there is nothing special about  $2 \times 2$  matrices; but Strang wrote the book. Anyway, the sum of two such matrices is another such matrix, and a scalar multiple of such a matrix is also such a matrix.
- The set  $F[-1, 1]$  of all functions from the interval  $[-1, 1]$  into the reals. (The text uses the set  $\mathbf{F}$  of functions on the whole real line. I’ve changed that to make the point that the domain really doesn’t make much difference.) We add two such functions, say  $f$  and  $g$ , and we multiply one, say  $f$ , by a scalar, say  $c$ , “pointwise”, i.e., at each value of  $x$  in  $[-1, 1]$ , the values of the resulting functions  $f + g$  and  $cf$  are given by the rule

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (cf)(x) = c(f(x)) .$$

This is of course the simplest addition and multiplication possible. Here’s a picture:



- The text uses  $\mathbf{Z}$  for the vector space with only one vector, the zero vector. It doesn't matter which space this vector is in, because they are all identical from a vector space point of view: The sum of the vector with itself is itself, all the scalar multiples of the vector are itself.

What we are really interested in at this point are subsets of vector spaces which are vector spaces in their own rights, with the addition and scalar multiplication inherited from the larger space. These are called *subspaces* of the larger space, and it turns out that, for a subset to be a subspace, all that has to be true is that

- the sum of two vectors in the subset is also in the subset, and
- all scalar multiples of a vector in the subset is also in the subset.

(Well, technically, you need one more thing: The subset can't be empty, because you need the zero vector there.)

So, for example:

- In  $\mathbb{R}^3$ , the set of vectors of the form  $(a, b, a + b)$  is a subspace, because
  - $(a, b, a + b) + (c, d, c + d) = (a + c, b + d, a + b + c + d)$ , so the sum of two vectors of this form is still of this form (the third entry is the sum of the first two:  $a + b + c + d = (a + c) + (b + d)$ ); and
  - $c(a, b, a + b) = (ca, cb, c(a + b)) = (ca, cb, ca + cb)$ .
- In any vector space  $V$ , the one-vector vector space  $\mathbf{Z}$  and  $V$  itself are subspaces.
- If  $\mathbf{v}$  is a vector in some vector space, then the set of all scalar multiples of  $\mathbf{v}$  form a subspace. In fact, the span (the set of all linear combinations) of any set of vectors in a vector space is a subspace — in fact, it is the smallest subspace that contains those vectors.
- The set of all  $2 \times 2$  matrices of the form  $\begin{bmatrix} a & b \\ b & a - 2b \end{bmatrix}$  is a subspace of  $\mathbf{M}$ . So is the set of all symmetric matrices:  $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ .
- The set of real-valued functions  $f$  on  $[-1, 1]$  that satisfy  $f(1/2) = 0$  is a subspace of  $F[-1, 1]$ . (There is nothing special about the  $1/2$  here — it could have been any number in  $[-1, 1]$  — but the  $0$  that follows it must be a zero; otherwise, the sum of two functions of this kind wouldn't be another function of this kind, so it wouldn't be a subspace.)

In view of one of the examples above, we can write down the first (kind of) subspace in which we are really interested:

**Definition.** For any  $m \times n$  matrix  $A$ , the *column space*  $\mathbf{C}(A)$  of  $A$  is the span of its columns, a subspace of  $\mathbb{R}^m$ .

Why do we care? Because, for any vector  $\mathbf{x}$  in  $\mathbb{R}^n$ ,  $A\mathbf{x}$  is a combination of the columns of  $A$ , with coefficients the components of  $\mathbf{x}$ . So  $A\mathbf{x} = \mathbf{b}$  has a solution exactly when  $\mathbf{b}$  is in  $\mathbf{C}(A)$ .

At this point, switch to the presentation for Unit 5.

*3.2. The Nullspace of  $A$ : Solving  $A\mathbf{x} = \mathbf{0}$*