Chapter 8: Applications

8.3. Markov Matrices, Population, and Economics

Most of this material is presented clearly enough in the presentation for Unit 11, but here is a bit of supplementary material.

The proof of the Perron-Frobenius Theorem of positive matrices given in the text has some gaps, so here is a complete proof.

Theorem. ***Perron-Frobenius Theorem for*** $A > 0$**.** *Suppose* A *is an* $n \times n$ *matrix with all (strictly) positive entries. Then the largest eigvalue* λ_{max} *of* A *in absolute value is (real) positive and has an eigvector with strictly positive entries.*

Proof. Consider the function f on the piece P of the hyperplane $x_1 + x_2 + \cdots + x_n = 1$ in $\mathbb{R}^n$ where all the x_i's are positive, given by

$$f(\boldsymbol{x}) = \min_i((A\boldsymbol{x})_i)/x_i = \min_i(\sum_j a_{ij}\frac{x_j}{x_i}) \ .$$

The minimum of a family of continuous functions is continuous, so f is continuous on P. For $\boldsymbol{x}$'s approaching an edge of P, where $x_i = 0$, say, the function $\sum_j a_{ij}(x_j/x_i)$ for that i approaches ∞, so the minimum, i.e., the value of f, is given by one (or more) of the other functions, which are continuous near that edge. So we can extend f continuously to the boundary of P and get a function f from $\overline{P}$, i.e., P together with its boundary, into $\mathbb{R}$.

A continuous function on a "compact" set like $\overline{P}$ always takes on its highest value (this is called the Extreme Value Theorem in Math 323), so we can pick $\boldsymbol{x}$ in $\overline{P}$ for which $f(\boldsymbol{x}) = t_{max}$ is highest. We claim this $\boldsymbol{x}$ is not on the boundary of P: Assume, by way of contradiction, that it is on the edge $x_i = 0$, and take the function $f_k = \sum_j a_{kj}\frac{x_j}{x_k}$ that gives the value of f near that point (so $x_k \neq 0$). Then $\partial f_k/\partial x_i = a_{ki}/x_k > 0$, so f is increasing in the direction of the positive x_i-axis. (In fact, it is increasing in the direction of all axes except x_k.) So $f(\boldsymbol{x})$ is not the largest f ever gets, and we have the contradiction. So f takes its largest value on P, not on the edge.

Next, we want to show $A\boldsymbol{x} = t_{max}\boldsymbol{x}$; so assume not. Now the strange definition of f was chosen so that $A\boldsymbol{u} \geq f(\boldsymbol{u})\boldsymbol{u}$ (i.e., the "$\geq$" holds in every component) with equality in at least one component, for every u in P; so our assumption that we don't have equality means we must have the strict inequality $(A\boldsymbol{x})_i > t_{max}x_i$ for some i . Now $A\boldsymbol{x}$ is probably not on the hyperplane that contains P, but the formula for f takes the same value on the whole ray from the origin through $A\boldsymbol{x}$, and that ray hits P somewhere, say at $\boldsymbol{y}$. Because $A\boldsymbol{x} - t_{max}\boldsymbol{x}$ has one strictly positive entry and <u>all</u> the entries of A are strictly positive, all the entries of $A(A\boldsymbol{x} - t_{max}\boldsymbol{x})$ are strictly positive; so we get that $(A\boldsymbol{y})_i > t_{max}y_i$ in every component, i.e., $f(\boldsymbol{y}) > t_{max} = f(\boldsymbol{x})$. But that contradicts our choice of $\boldsymbol{x}$. Therefore, $A\boldsymbol{x} = t_{max}\boldsymbol{x}$; i.e., t_{max} is an eigenvalue of A, with eigenvector $\boldsymbol{x}$ with all positive entries.

One last detail: Is it possible that some eigenvalue λ of A, maybe negative or complex, has absolute value greater than t_{max}?

(Interlude: The absolute value of a complex number $a + bi$ is $\sqrt{a^2 + b^2}$, its distance from the origin in the complex plane. We still have $|p + q| \leq |p| + |q|$ and $|pq| = |p||q|$, even for complex scalars p, q.)

The answer is no, because we can take absolute values and use the Triangle Inequality.: If $A\boldsymbol{z} = \lambda\boldsymbol{z}$ where λ and $\boldsymbol{z}$ may have positive entries, then

$$|\lambda||\boldsymbol{z}| = |A\boldsymbol{z}| \leq A|\boldsymbol{z}| \ ,$$

where $|\boldsymbol{z}|$, the vector with entries the absolute values of the entries in $\boldsymbol{z}$, is in $\overline{P}$, so $|\lambda| \leq f(|\boldsymbol{z}|) \leq t_{max}$. □

For our purposes, the point of this theorem is that, if a Markov matrix M has all nonzero entries, then it has a maximum eigenvalue, with a corresponding vector of positive entries. That maximum eigenvalue must be 1, because if $\boldsymbol{x}$ is an eigenvector corresponding to the eigenvector t_{max}, then for every vector we have

$$t_{max} \left[\begin{array}{cccc} 1 & 1 & \dots & 1 \end{array}\right] \boldsymbol{x} = \left[\begin{array}{cccc} 1 & 1 & \dots & 1 \end{array}\right] M\boldsymbol{x} = \left[\begin{array}{cccc} 1 & 1 & \dots & 1 \end{array}\right] \boldsymbol{x} \,.$$

So repeated left multiplications by M will produce a sequence of vectors approaching a steady state, no matter what vector we start with.

`At this point, switch back to the presentation for Unit 11.`