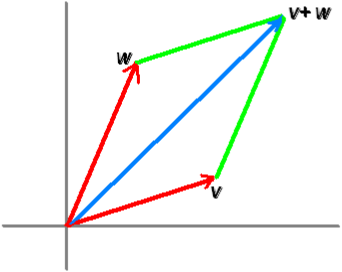
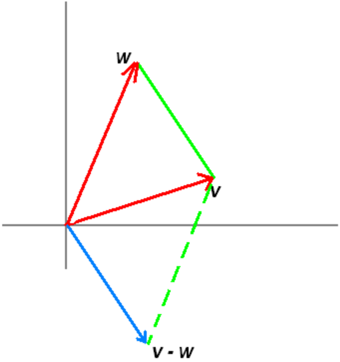
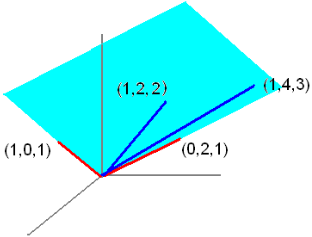
# Vector sum and difference



# Combinations

Combinations of $(1,0,1)$ and $(0,2,1)$:




$$(1,2,2) = 1(1,0,1) + 1(0,2,1)$$
$$(1,4,3) = 1(1,0,1) + 2(0,2,1)$$
$$(-1,2,0) = (-1)(1,0,1) + 1(0,2,1)$$

But $(1,2,1) \neq c(1,0,1) + d(0,2,1)$ because to make the first two components right, $c = 1$ and $d = 1$, but then the third component is wrong.

# “Dependent” vectors

A set of vectors is *dependent* if one of them is a lin comb (i.e., in the span) of the ones before it, and (of course) *independent* otherwise. [Technically, that's not quite right, but it's equivalent unless the first one in the list is the zero vector.]

So

- $(1, 0, 1)$ is independent.
- $(1, 0, 1)$, $(0, 2, 1)$ is independent.
- $(1, 0, 1)$, $(0, 2, 1)$, $(1, 4, 3)$ is <u>dependent</u>.
- $(1, 0, 1)$, $(0, 2, 1)$, $(1, 2, 1)$ is independent.

# Cauchy-Buniakowsky-Schwarz inequality

$$|\mathbf{v} \cdot \mathbf{w}| \leq ||\mathbf{v}|| \, ||\mathbf{w}||$$

**Proof:**

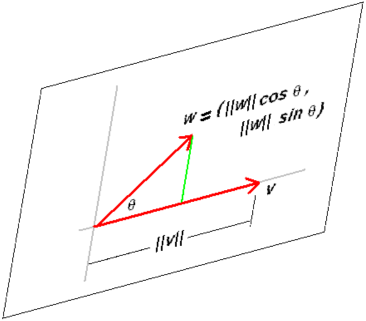
$$\begin{aligned}
||\mathbf{v}||^2 ||\mathbf{w}||^2 - (\mathbf{v} \cdot \mathbf{w})^2 \\
&= (v_1^2 + \cdots + v_n^2)(w_1^2 + \cdots + w_n^2) \\
&\quad - (v_1 w_1 + \cdots + v_n w_n)^2 \\
&= \sum_i v_i^2 w_i^2 + \sum_{i>j} (v_i^2 w_j^2 + v_j^2 w_i^2) \\
&\quad - \sum_i v_i^2 w_i^2 - 2 \sum_{i>j} v_i w_i v_j w_j \\
&= \sum_{i>j} (v_i w_j - v_j w_i)^2 \geq 0
\end{aligned}$$

# Geometry of the dot product

In fact, we could write the $\mathbf{v} \cdot \mathbf{w}$ in terms of $||\mathbf{v}||$, $||\mathbf{w}||$ and the cosine of the angle $\theta$ between them. Here's a proof that assumes that picking new axes doesn't change dot products (so it essentially begs the question):

In the plane of $||\mathbf{v}||$, $||\mathbf{w}||$ and the origin, pick axes so that $\mathbf{v} = (||\mathbf{v}||, 0)$; then $\mathbf{w} = (||\mathbf{w}|| \cos\theta, ||\mathbf{w}|| \sin\theta)$, so

$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &= ||\mathbf{v}||(||\mathbf{w}|| \cos\theta) \\ &\qquad + 0(||\mathbf{w}|| \sin\theta) \\ &= ||\mathbf{v}|| \, ||\mathbf{w}|| \cos\theta \ .\end{aligned}$$



# A better way

It would make better sense to show first that choices of new axes don't change dot products, but that is better saved for later in the course.

# "Orthogonal" — i.e., perpendicular — vectors

If $\mathbf{v} \cdot \mathbf{w} = 0$, then the cosine of the angle between them is 0, so that angle is right, so the vectors are perpendicular — we call that "orthogonal".

**Example:**

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 0$$

# Matrices

A *matrix* is a rectangular array of numbers.

**Example:** $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$

The (1,3) entry, i.e., entry in the 1st (horizontal) row and 3rd (vertical) column is $a_{1,3} = 0$.

There are 2 rows and 3 columns, so the matrix is $2\times3$.

# Multiplying a matrix by a vector

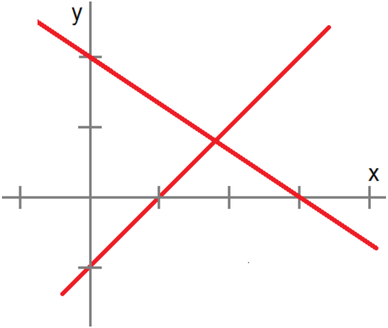
$$
\begin{aligned}
A\mathbf{v} &= \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \\
&= \begin{bmatrix} 1(3) + 2(4) + 0(5) \\ -1(3) + 1(4) + 2(5) \end{bmatrix} = \begin{bmatrix} 11 \\ 11 \end{bmatrix} \\
&= \begin{bmatrix} (1,2,0) \cdot (3,4,5) \\ (-1,1,2) \cdot (3,4,5) \end{bmatrix} \\
&= 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\end{aligned}
$$

Looking at the last version, we can see that $A\mathbf{v} = \mathbf{b}$ has a solution $\mathbf{v}$ exactly when $\mathbf{b}$ is a lin comb of the columns of $A$.

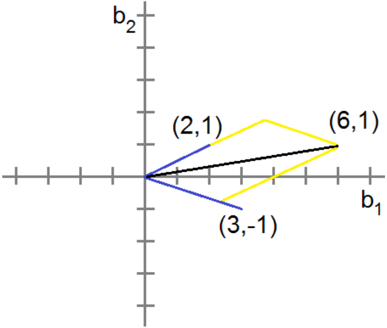
# Matrix equation $\iff$ system of lin eqns

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \iff \begin{cases} 2x + 3y = 6 \\ x - y = 1 \end{cases}$$

"Row picture"




"Column picture"



# $2 \times 2$ system, row picture version

- Graphs of both equations are lines.
- Usually, they are intersecting lines, so there is one common solution.
- But they may be the same line, so there are infinitely many solutions.
- Or they may be parallel lines, with no common solution.

## 2×2 system, column picture version

- Columns of the coefficient matrix are usually independent, so every column vector of constants is a lin comb of them in exactly one way.
- But they may be dependent, so their span is a line, and a column vector of constants on that line can be expressed in infinitely many ways, . . .
- . . . while a column vector of constants off that line cannot be expressed at all.

The coefficient matrix is *singular* if the columns are dependent, i.e., if there is not exactly one solution.

# Two versions, same result

For the system $\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$, there is not a unique solution if

- $cx + dy$ is a multiple of $ax + by$, i.e. $c/a = d/b$, i.e., $ad - bc = 0$.
- $\begin{bmatrix} b \\ d \end{bmatrix}$ is a multiple of $\begin{bmatrix} a \\ c \end{bmatrix}$, i.e., $b/a = d/c$, i.e., $ad - bc = 0$.

From earlier courses, you may recognize the number $ad - bc$ as the *determinant* of the coefficient matrix.

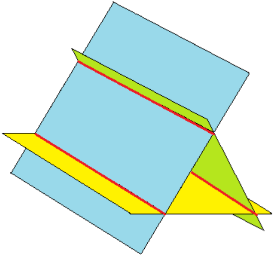
# 3×3 systems, row version

Graphs of the 3 equations are planes.

In most cases, the 3 planes meet in a point and there is only one common solution.

But all three planes may go through the same line, so there are infinitely many solutions.

Or two of the planes may be parallel, or each pair of planes may meet in a line, but the three lines are parallel; so there are no common solutions.



# Upper triangular system

With a "upper triangular" coefficient matrix (and all diagonal entries nonzero), we can always solve for the unique solution, by "back-substitution".

**Example:** $A = \begin{bmatrix} 3 & -1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$: Then for any $\mathbf{b} = (r, s, t)$, the equation $A\mathbf{v} = \mathbf{b}$ amounts to, for the unknown vector $\mathbf{v} = (x, y, z)$:

$$3x - y + z = r \qquad -y - z = s \qquad 2z = t$$

We must have $z = t/2$, so $y = -(s + t/2) = -s - t/2$, so $x = \frac{1}{3}(r + (-s - t/2) - t/2) = r/3 - s/3 - t/3$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3}r - \frac{1}{3}s - \frac{1}{3}t \\ -s - \frac{1}{2}t \\ \frac{1}{2}t \end{bmatrix}$$

# Upper triangular system

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3}r - \frac{1}{3}s - \frac{1}{3}t \\ -s - \frac{1}{2}t \\ \frac{1}{2}t \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & -1 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

# So, . . .

Can we turn every system into an upper triangular one?
No, but we can do a lot of them. But how?

Note that the following operations change a system into a new system that has the same solutions:

- Subtract from one equation a multiple of another equation.
- Reverse two equations.
- Multiply an equation by a nonzero constant.

We'll use the first one as much as we can to "pivot" on diagonal entries, only resorting to the second when the first "breaks down temporarily" (as the text puts it), and to the third only later.

# Pivot example 1

$$\begin{array}{rcrcrcr} 2x & + & y & & & = & 4 \\ x & + & 2y & - & z & = & 6 \\ 4x & & & - & 3z & = & 7 \end{array}$$

# Pivot example 1

$$\begin{array}{rcrcrcl} 2x & + & y & & & = & 4 \\ x & + & 2y & - & z & = & 6 \\ 4x & & & - & 3z & = & 7 \end{array}$$

"Pivot" on the coefficient 2 of $x$ in the first equation, using it to get rid of the $x$ terms in the other two equations: First subtract from the second equation 1/2 times the first: (The "multiplier" 1/2 comes from dividing the "pivot" 2 into the value 1 to be turned into 0.)

$$\begin{array}{rcrcrcl} 2x & + & y & & & = & 4 \\ & & \frac{3}{2}y & - & z & = & 4 \\ 4x & & & - & 3z & = & 7 \end{array}$$

# Pivot example 1

$$\begin{array}{rcrcrcr} 2x & + & y & & & = & 4 \\ & & \frac{3}{2}y & - & z & = & 4 \\ 4x & & & - & 3z & = & 7 \end{array}$$

Then subtract from the third equation $4/2 = 2$ times the first:

$$\begin{array}{rcrcrcr} 2x & + & y & & & = & 4 \\ & & \frac{3}{2}y & - & z & = & 4 \\ & & -2y & - & 3z & = & -1 \end{array}$$

# Pivot example 1

$$
\begin{array}{rcrcrcr}
2x & + & y & & & = & 4 \\
 & & \frac{3}{2}y & - & z & = & 4 \\
 & & -2y & - & 3z & = & -1
\end{array}
$$

Now pivot on the coefficient $3/2$ of $y$ in the second equation to get rid of the $y$-term in the third: Subtract from the third equation $-2/(3/2) = -4/3$ times the second:

$$
\begin{array}{rcrcrcr}
2x & + & y & & & = & 4 \\
 & & \frac{3}{2}y & - & z & = & 4 \\
 & & & & -\frac{13}{3}z & = & \frac{13}{3}
\end{array}
$$

# Pivot example 1

$$\begin{array}{rcrcrcr} 2x & + & y & & & = & 4 \\ & & \frac{3}{2}y & - & z & = & 4 \\ & & & & -\frac{13}{3}z & = & \frac{13}{3} \end{array}$$

The system is now upper triangular, and back-substitution gives the answer: $z = -1$, then $y = \frac{2}{3}(4 + (-1)) = 2$, then $x = \frac{1}{2}(4 - 2) = 1$.

# Pivot example 2

$$
\begin{array}{rcrcrcl}
x & + & 2y & - & z & = & 7 \\
2x & + & 4y & + & 3z & = & 4 \\
x & + & y & - & 3z & = & 9
\end{array}
$$

# Pivot example 2

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ 2x & + & 4y & + & 3z & = & 4 \\ x & + & y & - & 3z & = & 9 \end{array}$$

"Pivot" on the coefficient 1 of $x$ in the first equation, using it to get rid of the $x$ terms in the other two equations: First, subtract from the second equation 2 times the first:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ & & & & 5z & = & -10 \\ x & + & y & - & 3z & = & 9 \end{array}$$

# Pivot example 2

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ & & & & 5z & = & -10 \\ x & + & y & - & 3z & = & 9 \end{array}$$

Then subtract from the third equation 1 times the first:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ & & & & 5z & = & -10 \\ & & -y & - & 2z & = & 2 \end{array}$$

# Pivot example 2

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ & & & & 5z & = & -10 \\ & & -y & - & 2z & = & 2 \end{array}$$

The coefficient of the second variable $y$ in the second equation is 0, so "elimination breaks down temporarily", as the text puts it. But we can get back on track by reversing the second and third equations — we leave the first out of it, because it still has the $x$-term, so it would only complicate matters:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ & & -y & - & 2z & = & 2 \\ & & & & 5z & = & -10 \end{array}$$

# Pivot example 2

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 7 \\ & & -y & - & 2z & = & 2 \\ & & & & 5z & = & -10 \end{array}$$

And now the system is upper triangular and back-substitution gives the answer: $z = -10/5 = -2$, then $y = -(2 + 2(-2)) = 2$, then $x = 7 - 2(2) + (-2) = 1$.

# Pivot example 3

$$
\begin{array}{rcrcrcr}
x & + & 2y & - & z & = & 6 \\
2x & + & y & & & = & 4 \\
x & - & y & + & z & = & -2 \text{ or } 1
\end{array}
$$

# Pivot example 3

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ 2x & + & y & & & = & 4 \\ x & - & y & + & z & = & -2 \text{ or } 1 \end{array}$$

Subtract from the second equation 2 times the first:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & -3y & + & 2z & = & -8 \\ x & - & y & + & z & = & -2 \text{ or } 1 \end{array}$$

# Pivot example 3

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & -3y & + & 2z & = & -8 \\ x & - & y & + & z & = & -2 \text{ or } 1 \end{array}$$

Subtract from the third equation 1 times the first:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & -3y & + & 2z & = & -8 \\ & & -3y & + & 2z & = & -8 \text{ or } -5 \end{array}$$

# Pivot example 3

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & -3y & + & 2z & = & -8 \\ & & -3y & + & 2z & = & -8 \text{ or } -5 \end{array}$$

Now use the $-3$ in the second equation to get rid of the $y$-term in the third: Subtract from the third equation $-3/-3 = 1$ times the second:

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & -3y & + & 2z & = & -8 \\ & & & & 0 & = & 0 \text{ or } 3 \end{array}$$

# Pivot example 3

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ & & -3y & + & 2z & = & -8 \\ & & & & 0 & = & 0 \text{ or } 3 \end{array}$$

Now "elimination has broken down permanently", because the next pivot, the coefficient of third variable $z$ in the third equation is 0, and there is no equation below it to move up. But we can still read off the solutions to the system: The equation $0 = 0$ places no restriction on the variables, so we could take any value for, say, $z$, find the corresponding value for $y$ and then the corresponding value for $x$. So the initial system with last constant $-2$ has infinitely many solutions. But the equation $0 = 3$ is false no matter what values $x, y, z$ have, so the initial system with last constant 1 has no solutions.

# An example like some from the text

For which values of $a$ does elimination break down temporarily? How about permanently?

$$\begin{array}{rcrcrcr} ax & + & ay & + & 2az & = & a \\ ax & - & y & & & = & 3 \\ 2ax & + & 3ay & + & 2az & = & 4a \end{array}$$

- 
- 
- 

# An example like some from the text

For which values of $a$ does elimination break down temporarily? How about permanently?

$$\begin{array}{rcrcrcr} ax & + & ay & + & 2az & = & a \\ ax & - & y & & & = & 3 \\ 2ax & + & 3ay & + & 2az & = & 4a \end{array}$$

- $a = 0$: Breaks down permanently, because there can be no first pivot — the first column is all 0's. Hereafter, suppose $a \neq 0$.
-
-

# An example like some from the text

For which values of $a$ does elimination break down temporarily? How about permanently?

$$\begin{array}{rcrcrcr} ax & + & ay & + & 2az & = & a \\ ax & - & y & & & = & 3 \\ 2ax & + & 3ay & + & 2az & = & 4a \end{array}$$

- $a = 0$: Breaks down permanently
- $a = -1$: Breaks down temporarily, because the first two equations start $-x - y$, so after eliminating $x$ in the second and third equations, there is a 0 in the second pivot position. But we can then switch the second and third — the third now begins with $-y$ — to get a nonzero pivot.
-

# An example like some from the text

For which values of $a$ does elimination break down temporarily? How about permanently?

$$\begin{array}{rcrcrcr} ax & + & ay & + & 2az & = & a \\ ax & - & y & & & = & 3 \\ 2ax & + & 3ay & + & 2az & = & 4a \end{array}$$

- $a = 0$: Breaks down permanently
- $a = -1$: Breaks down temporarily
- For other $a$'s, eliminate $x$ in the second and third equations. We get

  $$\begin{array}{rcrcr} (-1-a)y & - & 2az & = & 3-a \\ ay & - & 2az & = & 2a \end{array}$$

  Then we can use $-1-a$ to eliminate $ay$ in the third equation, and the coefficient of $z$ in the last equation is $-2a - (-2a)a/(-1-a) = (2a+4a^2)/(-1-a)$. This is 0 when $2a + 4a^2 = 0$, i.e., $a = 0$ (which we handled) or $a = -1/2$: Breaks down permanently.

# An example like some from the text

For which values of $a$ does elimination break down temporarily? How about permanently?

$$\begin{array}{rcrcrcr} ax & + & ay & + & 2az & = & a \\ ax & - & y & & & = & 3 \\ 2ax & + & 3ay & + & 2az & = & 4a \end{array}$$

- $a = 0$: Breaks down permanently
- $a = -1$: Breaks down temporarily
- $a = -1/2$: Breaks down permanently.

So these three are the requested values.